Stiffness of restraint for steel struts

F. A. N. Al-Shawi

M. Banfi, *Arup Research & Development*

7. INTRODUCTION

It was interesting to read this paper and together with the previous work published in 1998 it is very useful in highlighting the importance of restraint stiffness. However, I think the conclusion that the restraint force should be increased from 1%–2.5% is too conservative and not justified by a closer examination of the numbers. Providing insufficient stiffness could also lead to bracing systems that are not stiff enough to restrain the member.

The use of 2.5% is similar to that in the Australian code AS4100-1998 that specifies a 2.5% restraint force and assumes that any system capable of taking this load will have sufficient stiffness. The older version (AS1250) did specify a stiffness required. However, 1% is also used by the AISC LRFD of 1999 which specifies a restraint stiffness required. The British Standard is perhaps unusual in not specifying the stiffness required although guidance is given in the SCI publication 093.

The purpose of the restraint is to reduce the effective length of the strut to a length based on the distance between the braces. If it is not stiff enough to do this, the strut may be under-designed. Both the AS1250 and the LRFD require factors on the minimum stiffness from an elastic critical analysis. The LRFD is based on the assumption that the strut is pin-ended between flexible restraints and the factor required is 2. The SCI also mentions the need for a stiffness significantly higher than the minimum required, with a criterion based on the strut stiffness that equates to a factor greater than 7.

Concerning the example given in appendix 1 of the paper. I have looked at the derivation of the strut values using both a 'hand' method and second-order analysis and I do not get the same answers as the design example. Further explanation is given below.

The example does, however, show the need to have adequate stiffness. If the stiffness is limited to the value given as being the minimum consistent with the 2.5% restraint force, the strut will not be restrained at mid-span. I carried out a buckling analysis that gives a critical buckling load for the strut of about 8600 kN. This equates to an effective length of 6.9 m and a strut capacity of about 5300 kN—that is, less than the

applied load. The actual stiffness applied (5.45 kN/mm) gives a critical load of about 16 000 kN. The mode still includes a displacement at mid-span but because of the end stiffness the equivalent effective length is close to 5 m. In practice this stiffness is lower than that which should typically be applied.

For a pin-ended strut, Timoshenko and Gere show that the minimum mid-span restraint stiffness required to give buckling in the two halves is such that $k_3 L/P_e$ is 16. With this stiffness, the $L/500$ initial imperfection and $P/P_e = 2$, the restraint force required is only 1.3%. Putting a factor of 2 on the stiffness reduces the restraint force to 1.1%. The 1% factor in BS 5950⁸ could be considered to be slightly low but if a slightly tighter imperfection limit, say 1 in 600, were assumed, it would be adequate. This would still be greater than the 1 in 1000 tolerance specified on elements.

The paper also suggests that the EC3 rules are consistent with the 2.5% force required but forgets that the EC3 rules are for a distributed restraint force. The formulae in the ENV version of EC3 appear to have been derived directly from considering a pin-ended strut with continuous restraint.

8. DERIVATION OF EQUATIONS FOR RESTRAINT FORCE AND CHECK ON DESIGN EXAMPLE

Using the notation in Fig. 1 of the paper and adding V_1 and V_2 as the reactions at ends 1 and 1 of the beam, then looking at the beam starting at the end, the differential equation for bending deflection is

$$z = A \sin \left(x \sqrt{\frac{P}{EI}} \right) + \frac{V}{P} x + \frac{M}{P} \left[1 - \cos \left(x \sqrt{\frac{P}{EI}} \right) \right] - \frac{P_{e0}}{P - P_e} \sin \left(\frac{\pi x}{L} \right)$$

where A is a constant and A , V and M have suffices 1 or 2 depending on the end.

There are seven unknowns: A , V and M at each end, plus F . The seven required equations are

- deflection equal to F/k_3 at spring (two equations)
- continuity of slope at spring
- slope at end equal to M/k (two equations)

- overall force equilibrium
- overall moment equilibrium.

These equations do not reduce to any reasonable expression unless the complexity of the problem is reduced (see below). They can, however, be solved using matrix methods in a spreadsheet. Using these methods, it appears that for the design example the spring stiffness of 1.41 kN/mm ($k'_3 = 3.43$) actually gives a restraint force of just below 2%. Also, to get a restraint force of 1%, a k'_3 value of 79.4 is required, rather than 114.6 as quoted in the design example. Two non-linear analyses were carried out to check this result. The results from both methods are given in Table 1. The differences between the approach given above and that of the original paper only apply to the design example. The figures in the paper are correct.

9. SIMPLIFIED EXPRESSIONS

For a symmetrical case—that is, mid-span restraint and $k_1 = k_2$, the following closed form solution can be derived

$$19 \quad \frac{F}{P} = -\frac{e_0}{L} \frac{1}{\left(1 - \frac{P_e}{P}\right)} \times \left[\frac{1 - \frac{k'_1 \pi P_e}{P \left(1 + k'_1 \pi \tan(\beta) \sqrt{\frac{P_e}{P}}\right)} \left(\frac{1}{\cos(\beta)} - 1\right)}{\frac{P}{k'_3 P_e} + \frac{\tan(\beta)}{2\pi \sqrt{\frac{P}{P_e}}} - 0.25 - \frac{k'_1 P_e}{2P \left(1 + k'_1 \pi \tan(\beta) \sqrt{\frac{P_e}{P}}\right)} \left(\frac{1}{\cos(\beta)} - 1\right)^2} \right]$$

where

$$\beta = \frac{\pi}{2} \sqrt{\frac{P}{P_e}}$$

and other symbols are as the paper. It can easily be seen how this simplifies if k_1 is zero or infinite.

For the simpler situation of a pin-ended strut the case for a restraint αL along the strut can be derived as

	$k'_3 = 3.43$		$k'_3 = 79.4$	
	Spreadsheet	Non-linear analysis	Spreadsheet	Non-linear analysis
M_1 : kNm	55.6	55.6	2.23	2.24
M_2 : kNm	204.7	204.8	8.20	8.23
V_1 : kNm	86.6	86.5	36.6	36.5
V_2 : kNm	56.8	56.7	35.4	35.3
F : kN	143.6	143.2	72	71.9

Table 1. Results from two non-linear analyses

$$20 \quad \frac{F}{P} = -\frac{e_0}{L} \frac{1}{\left(1 - \frac{P_e}{P}\right)} \times \left[\frac{\sin \pi \alpha}{\frac{P}{k'_3 P_e} + \frac{\sin \left(\pi (1 - \alpha) \sqrt{\frac{P}{P_e}} \right) \sin \left(\pi \alpha \sqrt{\frac{P}{P_e}} \right)}{\pi \sqrt{\frac{P}{P_e}} \sin \left(\pi \sqrt{\frac{P}{P_e}} \right)} - (1 - \alpha) \alpha} \right]$$

Author's reply

First of all I would like to thank Mr Banfi for his interest in the paper and his useful comments and the extra information he included from his own research.

The recommendation that $F/P\%$ is taken, generally, as 2.5 is intended to be used to cover common practical design cases. However, for economical design, use may be made of suitable design charts (similar to Figs 4, 5 and 6) which can be constructed from equation (14). Alternatively, the simultaneous equations (9)–(13) may be solved for any particular design situation.

Mr Banfi refers to Timosheko and Gere where k'_3 is equal to 16 (i.e. $k_3 = 16 \times 4109/10\,000 = 6.57$ kN/mm) which will result in a restraint force of 1.3%. A bracing system designed for strength to resist this restraint force is unlikely to have a stiffness of 6.57 kN/mm. This can be seen from the worked example for the two cases. Here the resulting bracing system designed for $F/P\% = 2.5$ has a stiffness $k_3 = 5.45$ kN/mm and for the case of $F/P\% = 1.0$ the resulting bracing stiffness $k_3 = 2.64$ kN/mm. Similarly, putting a factor of 2 on the stiffness gives $F/P\% = 1.1$ which will lead to a bracing system that is strong enough but not having sufficient stiffness.

Mr Banfi's concern about the difference in numbers he obtained from a different but similar approach is justified. The author agrees that, due to an arithmetical error, the values of k'_3 in the worked example should be 2.87 instead of 3.43 and 79.4 instead of 114.6. These values will result in $k_3 = 1.8$ kN/mm (which is still less than 5.45 kN/mm) and 32.63 kN/mm (which is still much greater than 2.64 kN/mm) for $F/P\% = 2.5$ and 1.0 respectively.

The calculation of strength and stiffness of the restraint system is essentially an iterative process and the use of $k_3 = 5.45$ kN/mm to give a critical load of 16 000 kN

represents the first cycle of calculations. For design purposes it is suggested to perform the calculations only once, by designing the bracing system for strength and then checking its adequacy for stiffness. Obviously, the larger the restraint force assumed, the more likelihood of getting adequate stiffness which

completes the design in one cycle, otherwise a second cycle may be required.

The analysis is based on a strut with initial out-of-straightness imperfection, and as the load is applied the displacement will increase because of the elasticity of the restraint, hence the inevitable mid-span displacement.

With regard to comparison with EC3 rules which give the restraint force, q per unit length, as uniformly distributed, it was assumed that when the restraint is located at a point, its value

may be taken as $qL/2$. In addition, the strut considered by EC3 is pinned at both ends while the general case considered in the paper has elastic supports. Therefore, the assumption made will lead to an approximation, but it does give the order of magnitude of the restraint force for the purpose of comparison with the results obtained in the worked example.

The equations derived by Mr Banfi are useful and may be used for the two special cases mentioned. The equations presented in the paper are intended to be used for any set of values of the relevant parameters.