Discussion

on papers published in the
Magazine of Concrete Research

Volume 30, Number 102 : March 1978

An equilibrium-compatibility approach to the determination of the strength of reinforced concrete rectangular beams under combined bending and torsion*

D. Stephen Sandegren and C. W. Yu

Contribution by Madhusudan Chakraborty†
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Dr Sandegren and Dr Yu must be congratulated for giving us an equilibrium-compatibility approach to the determination of the strength of reinforced concrete rectangular beams under combined bending and torsion. There are, however, several matters that must be questioned.

The inclination of the first crack at the soffit of the beam has been assumed to be the inclination of the failure cracks on all faces of the beam irrespective of (a) the ratio of the volume percentages of longitudinal to stirrup steel, (b) post-cracking change of the loading ratio prior to failure. This is unreasonable. For instance, according to their theory, a beam cracking under flexural moment only cannot resist any torsional moment, but experiments done to study the sequence of loading⁽¹⁾ show that sequence of loading has no appreciable effect upon the ultimate combined load. Further, horizontal legs of stirrups have been assumed to yield irrespective of the ratio of larger to shorter centre-to-centre dimension of stirrups. This is

contrary to the test observations made by Hsu⁽²⁾.

Though the authors have considered only mode 1 type of failure, no criteria have been given for the occurrence of such a failure. Further, they have suggested that the total force in the vertical legs of stirrups cut by the failure surface on either face be considered equal to that of the horizontal legs of stirrups. This may not be a reasonable approximation irrespective of the inclination of the failure crack and the ratio of the larger to the shorter dimension of the stirrup cage.

Dr Sandegren and Dr Yu, like others using the skewed bending approach, have neglected the torsional resistance of the concrete compression zone. An analysis based on both equilibrium and assumed compatibility of reinforced concrete beams under pure and combined torsion may not necessarily lead to an improved prediction of the ultimate load when the true compatibility conditions are not yet clearly known.

*Pages 3 to 10 of MCR 102.

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Contribution by V. G. G. Jayawardhana, BSc(Eng), MICE, CEng Polytechnic of the South Bank

The method proposed in this paper embraces a skewed bending approach and the following:

- (1) fixing the angle of cracks;
- (2) limiting the strains in the reinforcement;
- (3) an equation for applied bending moment and torque and internal moments.

The following discussion will show that (1) and (2) above are not pertinent to ultimate state, whereas (3) is not a sufficient condition for equilibrium, as there are components in the direction of the neutral axis. Strictly speaking, this method cannot be considered as applicable to any stage of the behaviour of the concrete beam, because first cracking of the beam and onset of yielding of reinforcement occur at different phases of the behaviour of the beam. The angle of cracks determined for the initial stage may not be applicable when the reinforcement attains yield strain.

St Venant's and Cowan's theories are applicable to pre-cracking and visco-elastic stages of loading respectively. Mohr's theory is applicable only to elastic conditions. Hence the assumption that the angle of the initial crack is the same as that of the final crack is irrelevant to ultimate state. It is also possible that the first crack may occur at a different angle on the vertical sides, as St Venant's theory predicts maximum stress at the centre of the wider faces of the beam. There are inelastic deformations before cracking, let alone after cracking.

The basis proposed by Dr Sandegren and Dr Yu can produce disastrously high theoretical values in beams with less reinforcement in the top layer than in the bottom when failure takes place with the formation of the compression zone in the soffit of the beam and yielding of the top layer of steel, though the first crack may appear at the soffit or vertical sides. This is possible at high T/M ratios.

If the loadings occur such that the final torque and/or moments are of opposite sense to their initial values, the actual behaviour is clearly contradictory to the basis proposed by the authors.

In the example (frame 4), the stirrups have reached no more than their yield strain and longitudinal steel has not reached its yield strain. Concrete in the compression zone has reached a very low strain compared with the strain at which concrete fails in compression. However, cleavage failure of concrete has been more often observed. If the stirrups are allowed a larger strain, longitudinal steel receives a larger strain than before, resulting in the theoretical values of T and/or M being larger. Therefore the assumption that the beam fails when yield strain is reached by either stirrups or longitudinal steel is arbitrary and irrelevant to the ultimate state.

The theory suggests that it is possible to obtain different strengths at the same final T/M ratio by varying the initial T/M ratio. This may not be possible in practice.

The equation for strength at failure as given in the paper can also be written as

$$M [\sin \beta + (T/M) \cos \beta] = A + B$$

or

$$M \sin \beta + T \cos \beta = A + B \dots (1)$$

where M , T and β are as defined in the paper;

A = moment of the longitudinal tensile forces about the centroid of the compression resolved perpendicular to the neutral axis;

B = moment of the stirrup forces about the centroid of the compression resolved perpendicular to the neutral axis.

The above equation as given in the paper could yield numerous conditions which are not applicable in practice.

First, consider the case where ignoring the increase or decrease of the depth of the neutral axis due to changes of stirrups and longitudinal steel affects the theoretical values slightly.

If, in the beam described in equation 1 above, the longitudinal steel is reduced so that A is now $A - a$ and if the stirrups are increased so that B is increased to $B + a$ and if this beam is now subjected to the same initial T/M ratio and M as before, then

$$M \sin \beta + T_1 \cos \beta = (A - a) + (B + a) = A + B \dots (2)$$

where T_1 is the torque in the latter beam.

Subtracting equation 1 from equation 2, we get

$$(T_1 - T) \cos \beta = 0$$

as $\cos \beta \neq 0$, $T_1 = T$.

Therefore the following deductions could be made from the method given in the paper:

reduction of stirrup steel can be offset by an increase of longitudinal steel to obtain constant T and M values;

reduction of longitudinal steel can be offset by an increase of stirrup steel to obtain constant T and M values.

If, in the beam described in equation 1, A is changed to A_1 and B is changed to B_1 and if the ratio of $(A_1 + B_1)/(A + B) = K$ and the same initial T/M ratio is maintained, then

$$M_1 \sin \beta + T_1 \cos \beta = A_1 + B_1 \dots (3)$$

Dividing equation 3 by equation 1, we get

$$\frac{M_1 \sin \beta + T_1 \cos \beta}{M \sin \beta + T \cos \beta} = K$$

If M_1 is made equal to KM , it can be shown that $T_1 = KT$.

Therefore another deduction from the method given in the paper is that, irrespective of whether stirrups or longitudinal steel are increased or decreased, the values of T will be changed to KT if M is kept at KM and vice versa when $A + B$ is altered to $K(A + B)$.

Even when the change of the depth of the neutral axis is significant, similar deductions could be arrived at. These predicted conditions are unrealistic and not supported by experimental evidence.

The comparison of the theoretical values with the experimental results is not as satisfactory as would be comparisons with some of the many test results available, preferably including some direct tests, as there could be errors in the assessment of torques and moments carried by the members of the statically indeterminate frame. Axial forces could also have been present. The test procedure adopted in finding the ultimate strength may result in values inconsistent and lower than the actual ultimate strength. Therefore it is difficult to say how appropriate are the comparisons of these test results with the other theories.

The sum of the contributions due to aggregate interlock, dowel action and the torsion taken by the concrete in the compression zone can sometimes be significant.

Contribution by A. S. Hall* and R. F. Staley† *University of New South Wales*

It is a pity that the authors do not state more clearly their purpose in conducting the tests reported. They calculate what they call the 'strength' of each beam as the values of moment and torsion at yielding of the longitudinal steel or the transverse steel, whichever yields first. These values may, of course, be what they are interested in but, in that case, it is not valid to compare such values with those obtained by other theories. Other investigators have been concerned with finding the maximum values of moment and torsion. After first yield, the member will accept more load until the second steel yields – unless concrete crushing supervenes, and this appears unlikely in these tests. So, obviously, the values calculated by the present theory will be lower than those calculated by other theories. It is not clear how the experimental values corresponding to initial yield were measured.

The authors do not make clear why they base their theory on a compatibility condition. The strength of under-reinforced beams can be computed quite well from equilibrium conditions alone. In the present theory, the value of the compression force, P_{cc} , is based on a linear stress distribution in the concrete, from which it follows that the concrete stresses are relatively low when the steel yields, i.e. the theory applies only to under-reinforced beams. Apart from this, the authors make no reference to the theories⁽³⁻⁶⁾ put forward in recent years, in which compatibility conditions have been incorporated.

The theory itself presents several features which call for comment. The proposal to predicate the failure surface on the direction of the principal strain at one point only, and that when the beam is still uncracked, is open to criticism. It follows from this proposal that the inclination of the failure surface is independent of the ratio of longitudinal to transverse steel, although this ratio plays an important part in all other theories and its importance is confirmed by tests. Incidentally, the formulae given on page 6 for the calculation of ϵ_x and ϵ_y are valid only if ϵ_n , which is at an angle β to the cross-section, is a principal strain and if the other principal strain is zero. This contradicts the previous assumption that the principal tension in the soffit is at an angle α to the cross-section.

We have applied the authors' theory to one of Hsu's beams tested in pure torsion – beam B8 of reference 7. The beam was 15×10 in. with one $\frac{1}{2}$ in. dia. longitudinal bar in each corner. Stirrups were $\frac{1}{2}$ in. dia. at a spacing of $2\frac{1}{4}$ in. For pure torsion, $\alpha = 45^\circ$. With $b = 10$ in., $h = 15$ in. and the neutral axis depth, x , assumed as 1 in., the authors' equations give $\beta = 14.74^\circ$. This leads to $\epsilon_y = 14.4\epsilon_x$. That is to say, the strain in the transverse steel is predicted as far greater than that in the longitudinal steel and a change in the value of x has little effect upon this conclusion. In the actual test, the longitudinal steel yielded first and its strain far exceeded that of the stirrups, which did not yield before the beam failed. This result could have been expected with such heavy transverse steel. In the present theory, β is indepen-

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dent of the ratio of longitudinal to transverse steel and this is not in accordance with much experimental evidence.

The forces in the vertical legs of the stirrups of the authors' beams are assumed to be equal to that in the bottom horizontal legs, although no basis is given for this assumption. Far from being 'reasonable' (page 7), this contravenes their compatibility condition, which would indicate vertical leg forces close to zero.

Dr Sandegren and Dr Yu have developed a theory on the assumption that the compression zone forms in the top face of the beam. They do not indicate how the theory might be modified in those cases where the compression zone forms at the side or at the bottom of the beam. Failures of this type have been observed (reference 5 of the paper).

In Figure 9, the authors' test results are compared

with a torsion-moment interaction curve proposed by Hsu and it is suggested that this curve might be used as a guide in design. However, it should be noted that Hsu's curve was based on tests of beams with no transverse reinforcement. It would have little relevance to normal beams.

Mitchell et al.⁽⁷⁾ have shown that, unless the stirrup spacing is restricted to $(b_1 + h_1)/4$ or 300 mm, beams in torsion may not develop their full potential strength but may fail by bursting at the corners. In designing beams to test a new theory, it would be advisable to detail stirrups to meet this requirement.

It is stated in the paper that the tests were carried out on 'statically indeterminate space frames'. A little more detail would have been helpful, particularly an indication of how the values of torsion and moment were actually measured.

Reply by the authors

We would like to thank Dr Chakraborty, Mr Jayawardhana, and Professor Hall and Mr Staley for their contributions to the discussion and to reply as follows.

CONDITION OF COMPATIBILITY

In order to take into account the contribution of longitudinal and stirrup reinforcement of different characteristics, the condition of compatibility must be considered. The assumption that all reinforcement intersected by a crack will simply yield will not produce acceptable results.

ORIENTATION OF CRACKS AND STRAINS IN STEEL REINFORCEMENT

According to the equilibrium-compatibility proposal, the angle of cracking, α , is determined by the elastic theory, on the assumption that the direction of crack will remain constant. This assumption is common to most treatises on the subject. Since the soffit of the beam is now in a cracked state, the exact effective cross-section is difficult to predict. Therefore any stress approach would be inadequate and hence the strain approach is considered more realistic for determining the steel forces. NA is the axis of rotation and the causal strain at E is in the post cracking stage. This strain occurs perpendicular to NA (neutral axis) and hence there is no strain along NA.

In Figure 6 of the paper, SM is the direction in which the causal strain exists and there is no strain in the direction SN perpendicular to SM . Therefore, strain along SP which is RQ/SR equals $PQ \cos Q/SR$.

But $PQ = \Delta x$ and $SR = x/\cos \phi$. Substituting, strain along SP becomes $\Delta x \cos^2 \phi/x$, which is (strain along x) $\times \cos^2 \phi$. It is possible to incorporate the effect of longitudinal and stirrup steel areas while calculating the bending and shearing stresses that fix the direction x of the first crack. However, the resultant expression will be very complex and such complication is not justified, as any gross error in estimating the forces in the vertical legs cut will only have a limited effect upon the total moment about NA.

OTHER MODES OF FAILURE

Modes of failure caused by varying T/M ratios and/or the top to bottom reinforcement ratio have not been considered in the proposal. However, it has been used with success to check cases of combined bending, shear and torsion where failure occurred with NA on the sides (reference 7 of the paper).

CARRYING CAPACITY BEYOND YIELD OF STEEL

In statically determinate mathematical models, yielding of the longitudinal or stirrup reinforcement or the crushing of the concrete will govern the failure of the section. Strain-hardening would, of course, increase the strength of the section after yield, but this is generally ignored for simplicity. Also, in the case of members subjected to bending torsion and shear, there could be some degree of indeterminacy involved in resisting the external load, in which case increase in resistance beyond the yield of reinforcement or crushing of concrete in the assumed model may be possible. This increase is generally small.

TORQUE—MOMENT INTERACTION

As discussed in reference 8 of the paper, skew bending theories, including the E-C method, generally do not yield satisfactory results near pure torsion conditions. This is one of the reasons for suggesting in the paper upper-bound values for T and M equal to pure strength values. This restriction also helps to limit the increase in the predicted moments as the longitudinal and web reinforcements are increased.

On the question of loading history, including sequence of loading, it is felt that more basic tests are required to be conducted exhaustively to predict the strengths.

EXPERIMENTAL SET-UP

Details of the experimental work are given in reference 7 of the paper. The set-up is too complex to be described here.

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