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Some tests on the yield criterion for a reinforced concrete slab*

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Dr Kwiecinski has carried out an interesting series of experiments on the yield criterion for reinforced concrete slabs. It is at first surprising to see that as α changes from 0° to 45° , although the recorded ultimate values of the moment m_n on the yield line decrease by 16%, the quoted experimental values of the ratio m_n/m increase by 19%, in nominally isotropic slabs. The difference arises from the allowance which must be made because the number of reinforcing bars crossing the fracture line is not a continuous function of α , but must be an integer. In slabs with only a small number of yielding reinforcing bars, the method of making this allowance is of some importance. Table I gives values of the ratio of recorded moment m_n to the expected moment, on three hypotheses about the properties of a steel bar crossing a fracture line (i.e. the extent of kinking at yield), with the bar numbers given in columns 4 and 5 of Dr Kwiecinski's Table 1. The first column of figures naturally differs slightly from Dr Kwiecinski's column 9, and it appears that the test results support the partial-kinking criterion.

However, the small number of yielding bars not only necessitates special consideration as above, but also leads to a rapid variation of strength across the slab. Thus with $\alpha = 45^\circ$, the number of bars crossing a line parallel to the simple supports varies from 8 to 10 in little more than 2 in., and is only effectively reduced to 7 if bond is lost on one of the bars, as in Figure 2. On the no-kinking hypothesis, the values which would have appeared at A in Table I for 8, 9 and 10 effective bars, are 1.042, 0.926 and 0.834 respectively; i.e. the strength of the slab varies by 21% in a

distance only 40% greater than the slab depth, so that the position of the fracture line is critical. Indeed, it may not be possible to apply the concept of bending moment in a relatively thin plate to the experimental slabs. Further, with concentrated loading, the applied moment m_n also changes rapidly across the slab, by approximately 6% per in. Dr Kwiecinski's calculations, and Table I here, are based on the maximum value of applied moment and the minimum strength of the slab, and so should give an upper bound on the true value of m_n/m on the fracture line, this bound coinciding with the true value only if failure occurs immediately under the concentrated load, in the region of minimum strength. Such reasoning may make possible a reconciliation between Dr Kwiecinski's results, which apparently support the partial-kinking criterion, and my tests⁽¹⁾ in which many more bars crossed the fracture line and the applied moment was constant over the span, and kinking of the steel bars was not significant.

It is interesting that the extent of kinking for $\mu = 1.188$ can be calculated from equation 10 of Dr Kwiecinski's theoretical paper⁽²⁾. Thus, when $\alpha = 45^\circ$, the bars should kink through 10° and, when $\alpha = 75^\circ$, through 23° . Such plastic kinking, across cracks of the

TABLE I: Values of m_n/m by various theories.

α	No-kinking hypothesis. (Johansen)	Partial-kinking hypothesis $\mu = 1.188$	Full-kinking hypothesis (Wood)
0°	1	1	1
15°	1.099	1.013	0.952
30°	1.113	0.967	0.847
45°	1.188 (A)	1.000	0.842

*Pages 135 to 138 of *Magazine* No. 52.

order of $\frac{1}{8}$ in. wide, should leave noticeable permanent deformation of the steel. It would be useful to know whether Dr Kwiecinski saw such deformation during or after his experiments, and at what stage of loading he estimates the kinking to occur. One of the difficulties with a partial-kinking hypothesis is that unless kinking either is always insignificant or occurs completely ($\gamma = 0$) and instantaneously when the slab first cracks, one would expect the kinking to increase progressively as cracks widen, so that approximate ideal-plastic behaviour would not be obtained, and general conclusions could not be drawn from the experimental ultimate moments.

In the theoretical paper, equation 10 essentially describes the properties of a set of yielding steel bars; one would not expect these properties to be affected by the presence or absence of an orthogonal set. Thus the $(90^\circ - \alpha)$ bars should obey the equation

$$\sin \delta = A \cos \alpha \dots \dots \dots (I)$$

with the same value of A . Equations 10 and I are only compatible with Dr Kwiecinski's equation 8, which follows from equation 6, when $\alpha = 1$; it would appear that the restriction of torsion m_{nt} to zero in all anisotropic slabs is misconceived.

Contribution by R. J. Lenschow and M. A. Sozen *University of Illinois*

Dr Kwiecinski has proposed a new yield criterion for reinforced concrete slabs^(2,3) based on partial kinking of the reinforcement, and reported experimental results in support. We would like to present some experimental evidence and a few opinions in opposition to Dr Kwiecinski's position.

Kinking of reinforcing bars

A direct investigation of the efficiency of bars inclined to the direction of the principal moment was carried out by Houbolt⁽⁴⁾ in 1942 at the University of Illinois. Houbolt tested slabs measuring 5 ft 3 in. $\times$ 5 ft 3 in. $\times$ 5.17 in. over a 5 ft span with loads applied at the third-points of the span as shown in Figure I. The $\frac{3}{8}$ in. round bars made angles of α_1 and α_2 with the direction of the span and were hooked at the ends; the yield stress was 50,200 lb/in². Fuller details of the 15 test slabs are given in Table II.

The calculated values of the moment are based on an orthodox interpretation of the efficiency of the reinforcement with kinking completely ignored. The comparison of the measured and calculated values do not indicate any significant kinking. This conclusion

is not surprising in view of the physical phenomena associated with kinking. Consider Figure IIa which illustrates the generally accepted concept of kinking. With a thread-like bar and a wide crack, not only partial but full kinking would be tenable if the bar does not tear out. However, the proportions shown in Figure IIb represent practical cases more closely. With those proportions, there may be some re-orientation of the bar but its effect is negligible.

Figure I: Test arrangement used by Houbolt⁽⁴⁾.

Figure II: Diagrammatic representation of reinforcement crossing a crack.

TABLE 11: Properties of specimens and test results of Houbolt⁽³⁾.

Specimen	Concrete strength f'_c (lb/in ²)	Orientation and spacing of reinforcement				Ultimate moment per unit width		
		α_1	s_1 (in.)	α_2	s_2 (in.)	calculated (lb × 10 ³)	measured (lb × 10 ³)	measured calculated
H1	5,130	0°	3	90°	6	8.3	8.6	1.04
H2	4,910	45°	3	45°	3	7.9	8.0	1.01
H3	3,800	30°	3	30°	3	11.5	11.3	0.98
H4	3,980	60°	3	60°	3	4.0	4.2	1.05
H5	4,020	30°	3	60°	3	8.0	8.6	1.07
H6	3,830	0°	3	45°	3	11.6	11.4	0.98
H7	3,780	45°	3	90°	6	4.2	3.8	0.91
H8	3,760	0°	3	90°	6	8.2	8.8	1.07
H9	3,820	45°	3	45°	6	6.1	6.3	1.03
H10	3,450	30°	3	30°	6	8.9	9.1	1.02
H11	4,210	60°	3	60°	6	3.1	3.1	1.00
H12	4,040	60°	3	30°	6	5.0	5.0	1.00
H13	3,880	30°	3	60°	6	7.1	7.3	1.03
H14	3,540	45°	3	0°	6	7.8	8.3	1.06
H15	4,380	45°	6	90°	6	2.1	2.0	0.96

$d_1 = 4.67$ in., $d_2 = 4.3$ in.

TABLE 111: Properties of specimens and test results of Lenchow and Sozen.

Slab	Concrete strength (lb/in ²)	Orientation and spacing of reinforcement with respect to the longitudinal axis				Inclination of the yield line with respect to the longitudinal axis	
		α_1	s_1 (in.)	α_2	s_2 (in.)	calculated	measured
B9	3,820	45°	3	—45°	1½	—71.6°	—72.0 ± 1.0°
B11	4,800	22½°	1½	—67½°	2½	75.6°	76.0 ± 1.0°
B12	5,170	67½°	3	—22½°	1½	—80.8°	—82.0 ± 1.0°

$d_1 = 3.56$ in., $d_2 = 3.31$ in.; two reinforcing bars of ½ in. diameter.
Yield stress 50,000 lb/in²

Twisting moment along the yield line

The crucial assumption in Dr Kwiecinski's approach is the inadmissibility of a twisting moment along the yield line. With reference to this point, we would like to cite the results of three tests from a recent series conducted at the University of Illinois. These slabs were subjected to a pure bending moment over a central span of 4 ft 6 in. with a special support arrangement such that no twisting moments could be externally imposed on the specimen. The properties of the slabs are described in Table III.

The yield lines which formed in these slabs were inclined to the axis of the span; there had to be twisting moments along them. Again, these results were not surprising since the inclination of the yield line could be predicted closely by minimizing the resultant moment along the span of the normal and twisting moments at the yield line.

We are confident that twisting moments can exist along yield lines in non-isotropically reinforced slabs and hope that Dr Kwiecinski will continue his work to look further into this matter.

Comments on Dr Kwiecinski's tests

It would have been a valuable addition to information on this subject if Dr Kwiecinski had reported the individual test results including the load-deflexion curves rather than averages. The condition that concentrated loads were used in these tests may lead to a load-deflexion curve *without* a flat-top portion even if the stress-strain curve for the steel exhibits a flat-top range⁽⁵⁾, because of an acute strain concentration at the load point. Dr Kwiecinski reports that the ultimate loads "were read by means of a maximum pointer". Could these readings have been taken when the reinforcement was in the strain-hardening range?

We appreciate the problems of bond and of trying to obtain a uniform reinforcement with a finite number of bars. The best solution appears to be the use of wide slabs, as large a number of bars as conveniently possible, and hooking the bars.

In conclusion we would like to record our appreciation of the clarity of Dr Kwiecinski's reporting, the explicitness of his premises, and the consistency of his conclusions with his premises.

Reply by the author

I am grateful to Messrs Morley, Lenschow and Sozen for their interesting contributions. Unfortunately, Dr Morley's Ph.D. thesis is not known to me so it is perhaps difficult to discuss the matter of reconciliation of our results in full. The difficulty with the partial-kinking yield criterion pointed out by Dr Morley, of which I was fully aware, is not so serious if one realizes that, as a yield line widens, progressive crushing of concrete in the neighbourhood of kinks results in an increase of the kinking angles γ and δ : while this very widening of a crack causes, at the same time (it being assumed for the moment that crushing is contained), a decrease of γ and δ . It seems possible that a balance between these two conflicting tendencies may be achieved. Thus, the perfectly plastic behaviour would be still obtained to within a reasonable accuracy. Obviously, the best answer would be furnished by tests in which the relationship between bending moment and plastic curvature would be measured in order to ascertain whether or not a more or less horizontal "plateau" is encountered. It would be most desirable to carry such tests out, with the steel bars spaced as closely as possible to avoid the additional effects discussed.

Dr Morley's equation I should read $\sin \delta = (A/\kappa) \cos \alpha$ (see equation 9 of reference 6), which makes all my formulae consistent for any value of the coefficient of orthotropy, not only $\kappa = 1$.

Dr Lenschow and Dr Sozen are thanked for citing valuable data and for making inspiring comments. I readily admit that a diameter of a reinforcing bar (its rigidity as such) in comparison with the opening of the crack has some considerable influence on its kinking, and hence on the ultimate bending moments. The situations reproduced in Figures IIa and IIb are limiting ones and it is felt that the truth lies somewhere between. It would be useful to carry out a number of tests, involving variable diameters.

As regards the twisting moments along the yield lines, I expressed my present opinion in the answer to Mr Dinnat and Mr Mills.* I should like only to emphasize that, when any non-Johansenian yield criterion is admitted, the moments at yield cease to obey Mohr's circle transformation and the situation must be very carefully studied (see, for example, reference 7).

Examination of the load-deflexion correspondence would undoubtedly yield some valuable information; unfortunately, these measurements were not taken in my tests.

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