Redistribution of moments in two-span prestressed concrete beams*

by S. K. Mallick, Ph.D.

Contribution by M. Tichý, C.Sc.

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In connexion with Dr Mallick's research, I would like to draw attention to a point not yet fully clarified regarding the terms "ultimate load", "redistribution coefficient", etc.

There are two possible ways of interpreting the test results, the difference between them being determined by the meaning attributed to the ultimate load value P_{ult} . This can be the collapse load when all critical sections have attained their ultimate moments ($P_{ult} = P_{coll}$). Alternatively, the ultimate load can be the load carried by the structure when the first plastic hinge forms ($P_{ult} = P_1$). Here, let us suppose that the terms "attaining the ultimate moment" and "formation of the hinge" are both denoting the same limiting state of stress (which is true in the case of structures prestressed by high-tensile steel with no discontinuity in the stress-strain diagram).

Now, which of the two loads (P_{coll} , P_1) is decisive from the stand-point of structural safety? Obviously, no structure will in practice be loaded to full redistribution. The occurrence of the first hinge, marked by more or less extensive crushing of the concrete, will, if observed, lead to immediate unloading and other measures. Naturally, cases may occur where P_1 will be only slightly less than P_{coll} , but in spite of this the value of P_1 will be the more important.

In dealing with the problem of redistribution, we must appreciate the real state of the tested structure by taking into consideration only the effective mechanical values (e.g. ultimate moments), not the supposed ones. If it is assumed, as mentioned above, that $M_{hinge} = M_u$, the failure load $P_f (= P_{ult})$ could never exceed the full redistribution load P_r . Of course, if a strain-hardening effect (in the stress-strain diagram of the reinforcement or in the moment-curvature diagram of the section) is present, $P_f > P_r$ may be observed. So, in evaluating the redistribution phenomena, measured values of ultimate moments must be put in the equilibrium equations determining P_r , and, of course, into any other equations (for example, Guyon's); only then can the importance of a certain theory be appreciated. From the results obtained by Dr Mallick, it appears that for the determination of P_r (full redistribution load, $P_r = P_{coll}$) the calculated ultimate values M_u were used. On the other hand, the failure load P_f

(Table 4) is in agreement with the measured ultimate moments M_m . The P_f and P_r values both represent a full redistribution load, measured and calculated respectively. Using the ratio P_f/P_r (Table 4), Dr Mallick compares the accuracy of formulae for the ultimate moment only, which, of course, can be controlled directly in Table 3. This fact is certainly evident from the values in the P_f/P_r column: some of them (beams P7, P16, T2) are greater than 1, which cannot be possible in the light of our previous considerations. A similar method was surely employed for determining P_f/P_n and P_f/P_G .

The results obtained by Dr Mallick are not invalidated by his approach, owing to the fact that a relatively large number of beams was tested and evaluated, so that the weight of discrepancies is not important. The resulting method is sound and gives a good picture of the behaviour of prestressed continuous beams. However, if a smaller number of beams is tested, the importance of the inaccuracies can be greater and can lead to confusing conclusions.

In conclusion I would like to propose that in all future tests certain "obligatory" data should be collected and published by the research workers, namely:

- load at the occurrence of the first hinge;
- collapse load (hinges in all critical sections);
- measured ultimate moments;
- measured cracking moments;
- secondary moments;
- rotation of the hinges.

In this way, co-operation would be achieved and this is of great importance, since the tests on continuous beams are rather expensive and only in a few cases is it possible to undertake them on such a large scale as Dr Mallick did.

Contribution by Sukumar Ghosh,

B.Tech(Hons), M.Tech.

(Messrs Chatterjee and Polk, Calcutta)

I would like to take this opportunity of presenting test results from four two-span beams similar to the rectangular beams in Group A of Dr Mallick's paper and to Professor Lin's with respect to loading. The details are summarized in Table I; a Riehle testing machine (40,000 lb range) on a 25 ft test bench was used. A spreader beam was used and reactions were measured. The notation is the same as that in the paper.

*Pages 171 to 183 of Magazine No. 42.

TABLE I: Details of beams.

Beam	Size (in.)	a/l	Elastic moment ratio E	F_{cu} (lb/in ²)	Number of 0.2 in. diameter high- tensile wires	Measured depth of cable (in.)		Mild-steel reinforcement under load		
						under load d_p'	over support d_p''	number and diameter (in.)	d_m' (in.)	length (ft)
B-1	4 × 8	0.37	1.67	5,480	4	5.6	6.40	2 × $\frac{3}{8}$	7.0	7
B-2	4 × 8	0.37	1.67	5,780	5	5.4	6.45	2 × $\frac{1}{2}$	7.3	7
B-3	4 × 6	0.27	2.37	5,100	5	3.4	4.35	—	—	—
B-4	4 × 6	0.27	2.37	5,100	4	3.8	3.50	2 × $\frac{3}{8}$	5.0	7

All the beams were provided with $\frac{1}{4}$ in. diameter stirrups at 4 in. centres between loads.

TABLE II: Comparison of moments and loads.

Beam	$\frac{M_{ml}}{M_{ul}}$	$\frac{M_{ms}}{M_{us}}$	$\frac{M_{Gl}}{M_{ul}}$	$\frac{M_{Gs}}{M_{us}}$	$\frac{P_f}{P_r}$	$\frac{P_f}{P_n}$	$\frac{P_f}{P_G}$
B-1	0.90	1.03	0.780	1.0	0.94	1.40	1.10
B-2	1.02	0.94	0.815	1.0	1.03	1.52	1.18
B-3	1.08	1.00	0.735	1.0	1.04	1.29	1.20
B-4	0.98	0.99	0.805	1.0	0.97	1.89	1.11

The comparisons in Table II support the findings of Dr Mallick.

A theoretical method of analysis requires the definition of the moment-curvature relation for the critical sections. Table III gives the ratios of experimental values of rotations at 90% and 70% of the ultimate moments. At 90% of the ultimate moment, the arithmetic means of columns 1 and 3 are 0.574 and 0.508 which compare favourably with 0.50 in the standard curve proposed by Y. Guyon. But the range of variation—from a maximum of 0.775 to a minimum of 0.29—is rather too wide to permit the general use of the mean values. Similar conclusions apply to state III at 70% of the ultimate moments: the individual values range from 0.063 to 0.425, and even the average values 0.268 and 0.150 are very different.

I have had the opportunity of studying Dr Mallick's complete report and have found similar variations; an attempt to fit simple curves to the experimental moment-rotation diagrams proved unsuccessful.

The discussion following Figure 14 of the paper show that rotations calculated as $Bl_p\theta_u$ assume very large values to ensure compatibility and Dr Mallick introduces an extra concentrated rotation at the first hinge in setting up the compatibility conditions. His expression for ψ_L is empirical but I think the underlying suggestion that the moment-rotation relations are likely to be different at the respective hinge sections (p. 178) is sound, as Table IV shows.

TABLE III: Experimental moment-rotation relation.

Beam	1	2	3	4
	Under load		Over support	
	$\frac{\theta_{2m}}{\theta_m}$	$\frac{\theta_{3m}}{\theta_m}$	$\frac{\theta_{2m}}{\theta_m}$	$\frac{\theta_{3m}}{\theta_m}$
B-1	0.500	0.296	0.635	0.220
B-2	0.435	0.195	0.412	0.063
B-3	0.585	0.155	0.290	0.149
B-4	0.775	0.425	0.695	0.168
Average	0.574	0.268	0.508	0.150
Standard values (Guyon)	0.500	0.170	0.500	0.170

θ_m = probable ultimate rotation (measured).

θ_{2m} = rotation at 90% measured ultimate moment.

θ_{3m} = rotation at 70% measured ultimate moment.

In Table V, I have listed the values of φ_D and corresponding ψ_L for the support sections calculated on different considerations. For φ_{De} , I have used failure strains of 0.45% and 0.30% at the support and loaded sections respectively. Table IV gives the average values as 0.628% and 0.445%, i.e. the strain at the support is roughly 1.5 times that at the loaded section. Numerically ψ_{Le} is least; this is also approximate.

It seems imperative to have a close assessment of the rotation actually taking place in the plastic zone of a beam. This involves measuring strains all along the length of the beam or at least over a large part of it and then plotting the rotations given by e_c/n at different sections, followed by integration as $(e_c/n)dx$. This aspect would perhaps be an interesting field of investigation and was suggested by Dr Mallick during the discussions I have had with him.

TABLE IV: Strains and rotations at failure. All strains are in % and rotations in rad/in. $\times 10^{-3}$.

Beam	e_{ms}	e_{ml}	θ_{us}	θ_{ul}	θ_{ms}	θ_{ml}	$\frac{\theta_{us}}{\theta_{ul}}$	$\frac{\theta_{ms}}{\theta_{ml}}$
B-1	0.51	0.630	2.18	1.89	4.10	2.7	1.16	1.52
B-2	0.80	0.420	1.94	1.41	8.00	2.3	1.38	3.48
B-3	0.70	0.435	1.83	1.94	5.70	2.9	0.95	1.97
B-4	0.50	0.295	2.38	1.62	4.75	2.0	1.47	2.37

Average $e_{ms} = 0.628$ Average $e_{ml} = 0.445$

TABLE V: Distributed and local rotations at support. All rotations are in rad $\times 10^{-3}$.

Beam	1	2	3	4	5	6	7
	Distributed rotations φ_D			Local rotations ψ_L			
	φ_{DG}	φ_{DR}	φ_{De}	ψ_{LG}	ψ_{LR}	ψ_{Le}	$Eh\theta_u$
B-1	6.27	13.6	—	12.9	18.8	—	14.6
B-2	5.60	10.3	6.00	9.0	10.6	6.0	12.8
B-3	4.69	11.2	5.58	14.4	19.0	14.6	13.0
B-4	5.35	11.5	7.70	11.4	22.2	12.6	16.9

 φ_{DG} — by Guyon's method. φ_{DR} — from experimental moment-rotation curves. φ_{De} — using failure strain 0.45% and 0.3% at support and loaded sections respectively.

Contribution by L. N. Sastry, M.Tech. (Indian Institute of Technology, Kharagpur)

Dr Mallick's tests indicate clearly that full redistribution is a possibility in two-span beams; applied to three-span beams, similar to Macchi's, the proposed method predicts that over 90% M_{us} is likely to be attained. In these exploratory tests, the reactions were not measured and Figure I shows the development of cracks in beam 9, which is a hard case.

In the current tests (Table VI), the reactions are being measured and the results of one of them are given in Table VII.

Figure II shows the support section of Beam B₇.

The slope of the beam between the support and the load was found to be 1.75 in. in a length of 24 in., which is quite considerable.

Dr Mallick does not hide the approximation in his proposals for the distributed term; he has used $Bl_p\theta_u$ (θ_u corresponding to a failure strain of 0.35%) as a simplification. Figure III shows the measured rotation at different sections of the beam with those given by Guyon's curves for direct comparison. The rotations are given in Table VIII. The compatibility equation suggested by Guyon has been used.

TABLE VI: Details of beams.

No.	a (ft)	b (ft)	d_p' (in.)	d_p'' (in.)	F_{cu} (lb/in ²)	E	P_f/P_r	P_f/P_G	Redistribution coefficient d		
									test	Guyon	Mallick
1*	8.0	8.0	3.0	4.4	4,900	2.33	0.954	1.160	0.94	0.720	0.920
2	7.0	10.0	3.5	3.5	7,000	1.90	0.946	1.070	0.900	0.770	0.970
3†	8.0	8.0	3.5	3.6	6,250	2.33	1.005	1.180	1.000	0.726	0.962
4	6.0	12.0	4.0	4.0	5,250	1.67	1.050	1.060	1.000	0.815	0.980
5‡	8.5	7.0	3.0	3.0	5,500	2.63	0.984	1.135	0.965	0.730	0.990
6	7.5	9.0	3.5	3.5	5,830	2.12	0.894	1.015	0.790	0.760	0.975
7	8.0	8.0	3.5	3.5	5,830	2.33	0.970	1.110	0.940	0.750	0.940
8	6.0	12.0	3.5	3.5	6,500	1.67	1.080	1.190	1.000	0.825	0.983
9	8.0	8.0	3.5	3.5	6,500	2.33	1.094	1.220	1.000	0.760	0.970

* At the loaded section, the measured width was 3.5 in.; † prestressed with 4 wires; ‡ straight cable.

All beams are 4 × 6 in. prestressed with five 0.2 in. high-tensile steel wires by Gifford-Udall system and grouted.

At the loaded section, d is unity—the proportion of M_{us} that would enable the test load to be carried. Only the results of No. 6 are near to Guyon's, but 1, 3, 5, 7 and 9 represent harder cases.

TABLE VII

a	b	dp'	dp''	F _{cu}	P _f /P _r	P _f /P _g	Redistribution coefficient d	
							test	Guyon
8	8	4	4	5,000	1.00	1.13	1.00	0.76

TABLE VIII

Guyon	Test
$0.92 \alpha_1' + \alpha_2' = 17.45 + 5.20$ $= 22.65$ $\alpha/2 = 5.20$	$0.92 \alpha_1' + \alpha_2' = 11.80 + 6.68$ $= 18.48$ $\alpha/2 = 17.93$

The compatibility condition $0.92 \alpha_1' + \alpha_2' = \alpha/2$ is very nearly satisfied at the failure of the beam.
 $= \alpha/2$ is very nearly

Figure I

Figure II

Figure III

**Contribution by N. F. Somes, M.Sc.(Eng),
A.M.I.C.E., A.M.I.Struct.E.
(Cement and Concrete Association)**

Dr Mallick's paper is a welcome contribution to existing knowledge on this subject.

For the loading he has employed, Dr Mallick has found that the total rotation in the first plastic zone in tests is considerably greater than that calculated on the basis of Guyon's equation:

$$\Sigma \Delta \psi = B l_p \theta_u$$

where

l_p = length of plastic zone

θ_u = rotation associated with the maximum moment

B = factor dependent upon the shape of the bending moment along the plastic zone and the crack spacing.

Guyon⁽¹⁾ dealt analytically with several shapes of moment penetration and for each tentatively put forward values of factor B . For the angular penetration shown in Figure IV he suggested $B=0.33$. Such a shape is somewhat hypothetical since even a point loading is applied over a finite length giving rise to a flat at the peak of the diagram.

In Dr Mallick's tests the point of contraflexure is located only some 24 in. from the central support and the modifying effect of what is seemingly a 4 in. steel support plate will be significant. In this case 0.32 is thought to be too low a value to take for B .

Investigations into moment-rotation characteristics for prestressed beams are currently being carried out by the Cement and Concrete Association with 11×6 in. rectangular sections and a test span of 110 in. For 4 beams loaded centrally through a 3 in. wide steel plate the value of B ranged between 0.65 and 0.8 for the maximum moment condition. In each case crushing began some 1–3 in. from the edge of the plate.

What are the effects of confinement by the plate? It is known that transverse compression considerably increases the longitudinal compressive strength of

concrete. Balmer⁽²⁾ with cylinders under triaxial stress reported that an active transverse stress p_o increased the strength p in longitudinal compression such that

$$p = U_{cyl} + 5.4 p_o$$

a result similar to that obtained by Akroyd⁽³⁾. In Dr Mallick's tests the vertical compression under the loading plates will be considerable.

The following information would therefore be a useful addition to the paper:

details of the loading plates and where in general crushing occurred;

the points with respect to which M_{ml} and M_{ms} were calculated;

some comment on the general agreement between observed compression zone depth and that calculated on the basis of the stress-block coefficients of Figure 5.

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Reply by the author

Mr Tichý is right in pointing out that P_l may be decisive from the safety point of view. My investigations dealt with the extent of redistribution that may take place before collapse, and in the conclusions reference has been made to serviceability as well.

The ratio P_f/P_r does not always compare the formulae for the ultimate moments. In beam P6, redistribution was far from complete (M_{ms}/M_{us} in Table 3 is 0.82) but P_f/P_r is 0.92 (Table 4), i.e. the observed load is lower by 8% only. That this ratio is greater than 1 for three beams is, perhaps, of little importance; the computed ultimate moments controlling the value of P_r can be no more than assessments based on several assumptions. It may also be observed that for beams P4 and P11, the ratios are less than 1 even though all critical sections had failed in the tests (Figures 8 and 9).

I have been associated with the tests carried out by Messrs Ghosh and Sastry and am pleased to find that they have communicated their findings which lend support to the principal conclusion from my tests: that redistribution is a possibility in all but the exceptional cases.

The rotation diagram given by Mr Sastry (Figure III) is interesting and informative; it provides a clear

Figure IV

explanation for the discrepancy between the test results and the values obtained by the application of Guyon's theory. I have tried to indicate a simple method of computation; the approximations have been stated and Mr Ghosh has elaborated alternative methods for the evaluation of the local rotation. The method appears to give close agreements for the tests on three-span beams as well (Table VI) but the rotations used in the calculations can be said to be "fictitious" in the sense that they do not represent the real magnitudes.

The data in Table VIII or Figure III are likely to provide a rational basis for computations based on the rotations actually taking place at the hinge sections. It may be possible to express the measured rotations α_{1M} , α_{2M} , α_{3M} (M = measured) as $F_1\alpha_{1G}$, $F_2\alpha_{2G}$, $F\alpha_G$ (G = Guyon's method), the multiplying factors F_1 , F_2 , F , etc. being deduced from diagrams similar to Figure III for different types of beams as the ratios between the measured and the computed rotations.

The deflexion of $1\frac{3}{4}$ in. in 24 in. observed by Mr Sastry confirms a conclusion in the paper about the rotation capacity of prestressed concrete sections.

I agree with Mr Somes about the value of B being greater than 0.32 when the widths of the plates are allowed for; these plates change the triangular penetration to trapezoidal. In my tests, 4 in. plates were used for which the values of B_I and B_{II} for the two hinges in beam P4 would be as given below for the state of full redistribution. The data in Figure 17 and Table 9 have been used in the following calculations.

Support section

$l_p'' = 8$ in.; penetration is triangular over 6 in. and parallel over 2 in. $\varphi_s = (0.32 \times 6 + 1.00 \times 2) \theta_{us} = 3.92 \theta_{us} = B_I \times 8 \times \theta_{us}$ giving $B_I = 0.489$, and $1.36 \varphi_s = 0.489 \times 8 \times 2.23 = 11.8 \times 10^{-3}$ rad compared with 7.75 in Table 9.

Beneath load

By considering triangular penetration over 40 in. and parallel over 4 in., B_{II} is found to be 0.382 and $\varphi_I = 45.0 \times 10^{-3}$ rad compared with 36.5 in Table 9. Figure III also shows B as greater than B_{Guyon} at the first hinge and smaller at the second.

These modifications may lead to a slightly larger degree of redistribution theoretically but the extent observed in the tests would not be fully explained. In connexion with Mr Somes' tests on simple beams, I would like to mention that data from simple beams perhaps cannot be used for continuous beams in which there is more than one critical section and the moment-curvature relationships may vary from one section to another in a plastic zone.

The influence of the widths of the plates will become appreciable when the plastic zones are small. The confinement due to the plates is somewhat similar to the binding effect of closely spaced stirrups leading to an increase in the strain capacity. This has been mentioned in the complete report of the investigation in the discussion of the concentration of rotation at the first hinge.

The answers to Mr Somes' other queries are as follows. M_{ml} and M_{ms} were calculated at the centre-lines of the load and the central support respectively. In almost all beams, failure was complete at all critical sections and crushing occurred within the gauge sections as shown for two beams in Figures 8 and 9. The depth of compression could not be measured but can be determined from the data in Table 5. The following illustration is for beam P4.

At support

$\theta_{us} = 2.23 = 3.5/n_{rs}'$ or $n_{rs}' = 1.57$; $\theta_{ms} = 3.5 = 3.0/n_{rs}''$ or $n_{rs}'' = 0.86$ in.

Beneath load

In the same way, values are $n_{rl}' = 1.3$ in. and $n_{rl}'' = 1.43$ in.

The probable values (n_{rs}'' or n_{rl}'') cannot be compared directly with n_{rs}' or n_{rl}' obtained on the basis of Figure 5, as the measured ultimate strains and rotations were different from the value of $e_{cr} = 0.35\%$ assumed for the stress-block and the rotations θ_u calculated from it.

I am grateful to Messrs Tichý, Ghosh, Sastry and Somes for their discussions on the paper. Mr Tichý's proposal for certain obligatory data to be published by research workers is very laudable.