**Analysis of the behaviour of axially
compressed stringer-stiffened
cylindrical shells**

A. C. WALKER & S. SRIDHARAN

Dr C. P. Ellinas and Dr J. G. A. Croll, University College London

The Authors are to be congratulated in their aim of developing simple theoretical models for interpreting the behaviour of a much studied but incompletely understood problem of shell buckling. The theoretical modelling of the axial load buckling of stiffened cylinders, in common with the special case of isotropic cylinders, has not yet reached a practically satisfactory state. The increased analytical and numerical sophistication that has so often characterized recent theoretical developments has seldom led to significant improvements in the understanding of the physical process which determines the buckling of this important class of shell component. The Authors' underlying philosophy of attempting to define conceptually simple models which encapsulate the important physical characteristics of the buckling process is timely. However, there appear to be significant differences between our own attempts at formulating similar theoretical lower bounds for axial load buckling of orthotropic (narrow panelled) cylinders.²⁹ Also a number of approximations in the Authors' theoretical modelling of their experimental end boundary supports may have significant effects on the level of reliability claimed (§ 67) for their theoretical model.

75. We share with the Authors the observation that significant non-linearity of buckling always involves changes in the distribution of membrane stiffness (or energy). In many cases of shell buckling this appears to involve the shedding of the initial linear membrane strain energy as non-linear post-buckling progresses.^{21,30} However, the case of axially loaded isotropic cylinders is one of the pathological cases for which such an interpretation provides an inadequate representation of the great bulk of empirically available data. To obtain practically useful explanations of the observed axial load buckling of isotropic cylinders requires an elaboration of the basic notions of unstable buckling³¹ which would also appear relevant to the case of axial load buckling of orthotropic cylinders.²⁹

76. To illustrate the nature of this difference in approach consider a classical analysis of the Authors' shell model UC2. For the sake of illustration it is assumed that the end supports remain effectively rigid up to the classical critical load in which case the spectrum of critical loads is shown by the continuous line

in Fig. 12(a). For critical deformations into each of the circumferential wave numbers n , Fig. 12(b) summarizes all the energy terms contributing to the stabilization of the shell. The contributions from bending and membrane energy show much the same patterns as in other shell buckling problems, i.e. as the circumferential wavelength decreases (n increases) the decrease in membrane strain energy is accompanied by a monotonic increase in bending strain energy. However, unlike other shell buckling problems the axially loaded cylinder has an additional, and crucial, energy contribution to the shell's resistance to buckling. This additional stiffness^{29,31} may be reinterpreted as being part of the non-linear membrane strain energy. It derives from the product of the axisymmetric prebuckling strain, caused by the Poisson bulging of the axially loaded shell,

Fig. 12. (a) Critical stress spectra for shell UC2; (b) positive energy contribution spectra in the classical analysis for shell UC2

with the non-linear circumferential stress associated with the critical (or buckling) mode. However, it is these non-linear circumferential stresses that are annihilated by a coupling between this periodic critical mode and an axisymmetric mode of half the axial wavelength.^{29,31,32} Hence one characteristic of this post-buckling behaviour of the axially loaded cylinder is that the contribution to the stabilization of the shell, shown dotted in Fig. 12(b), would be eliminated at very small critical deformations. A lower bound to this non-linear coupling is then obtained by eliminating this particular non-linear membrane energy term from a critical load analysis of the shell model. For the isotropic cylinder this reduced stiffness modelling has been shown to provide realistic lower bounds for shells containing imperfection levels reminiscent of prototype civil engineering shells.³¹ Moreover, the model is found to predict closely the modes observed to trigger this buckling.^{31,33} These triggering modes are in sharp contrast with the familiar diamond shaped modes characterizing the post-buckling state. The Authors' equivalent column model, where the linear membrane strain energy is neglected, would lead to lower bound predictions that in the case of the isotropic cylinder would always be 50% of the classical critical stress, which is certainly not a lower bound for available experimental data.³¹ Moreover, for the isotropic shell the Authors' approach, like that of the classical analysis, would be indeterminate so far as the prediction of the buckling mode is concerned.

77. There appear to be good reasons for supposing that a similar coupling occurs in orthotropically stiffened shells. An analysis based on a shell model from which this component of the non-linear membrane energy has been eliminated would, for the Authors' shell UC2 with clamped ends, predict the critical load spectrum shown by the broken line in Fig. 12(a). A lower bound of the elastic buckling for this shell is approximately 46% of the minimum classical critical load. In contrast a model from which the linear (linear in the sense of being associated with linear critical stresses and strains) membrane strain energy has been eliminated, as proposed by the Authors, would take the form of the dotted spectrum in Fig. 12(a). For the Authors' modelling the lower bound would be as high as 85% of the minimum classical critical stress and would occur at a different wavelength from our own reduced stiffness lower bound.^{29,31} It would appear that such large differences in the predicted lower bounds to imperfection sensitivity should be able to be resolved in future carefully controlled elastic buckling tests of narrow panelled stiffened cylinders. Do the Authors feel it would be reassuring to the designer being urged in § 67 to adopt their approach if the relevance of the equivalent column method could be established at least for the special case of the isotropic cylinder?

78. In the interpretation of experimental buckling behaviour the Authors consider (§ 48) that for the narrow panelled shells the support arrangements initially corresponded to a fully clamped condition. The displacement functions (§ 37) were accordingly chosen to model such a classical clamped (C4) boundary condition for which

$$u = v = w = \partial w / \partial y = 0 \quad (20)$$

In § 49 it is suggested that a yielding of the stiffener outstand is followed by a localized plastic mechanism forming at the ends of the stringers with the consequence that the end conditions are reduced from fully clamped to a classical simple support. The displacement functions used to model this reduced stiffness boundary (§ 40) are those of the SS3 variety for which

$$n_x = v = w = m_x = 0 \quad (21)$$

79. Although loss of the flexural rotational restraint might seem plausible through such a mechanism, it is surprising that the clamping devices connecting the skin of the shell to the rigid end loading plates should also have lost their capacity to transmit the periodic axial stresses associated with the buckling deformations. If these end clamping devices remain effective would not the correct end boundary restraint be more closely related to the SS4 conditions

$$u = v = w = m_x = 0 \quad (22)$$

than to the SS3 conditions of equation (21)? For each of the models in Table 4, analyses based on the exact Donnell strain-displacement relations,²⁹ and including prebuckling flexural deformations, show that the critical load spectra are highly dependent on whether the SS3 or the SS4 boundary applies. Table 7 shows the minimum critical stresses and associated wave numbers for the case of the kinematic and static constraints (20)–(22) applied at the centroid of the orthotropic section. The critical stresses and modes are strongly dependent on

Table 7. Effects of boundary conditions on the minimum classical critical stress and corresponding wave numbers*

Shell	SS3 $\sigma/E \times 10^{-3}$	SS4 $\sigma/E \times 10^{-3}$	C4 $\sigma/E \times 10^{-3}$
UC2	2.77(8)	3.74(9)	5.84(9)
UC5	5.03(7)	5.21(8)	15.2(10/11)
UC6	1.97(9)	2.95(10)	3.59(10)
UC7	2.83(8)	3.14(9)	8.22(10)
B1	2.07(7)	2.64(8)	5.38(8)
B5	2.05(7)	2.40(8)	5.81(8)
UC9	1.93(9)	2.36(10)	5.17(10)

* Figures in brackets represent the circumferential wave numbers n corresponding to the minimum critical stress.

Table 8. Effects of kinematic and static eccentricity on classical critical stress*

Shell	SS3 $\sigma/E \times 10^{-3}$		SS4 $\sigma/E \times 10^{-3}$	
	$Z_p = 0.0$	$Z_p = -1.0$	$Z_p = 0.0$	$Z_p = -1.0$
UC2	2.77(8)	2.38(8)	4.82(9)	3.98(9)
UC5	5.03(7)	3.03(9)	7.88(9)	3.82(12)
UC6	1.97(9)	1.78(9)	3.40(10)	3.17(10)
UC7	2.83(8)	1.97(9)	4.83(10)	2.94(11)
B1	2.07(7)	1.69(7/8)	3.99(8)	3.05(9)
B5	2.05(7)	1.56(7/8)	3.94(8)	2.71(9)
UC9	1.93(9)	1.50(9/10)	3.47(10)	2.48(11)

* Figures in brackets represent the circumferential wave numbers n corresponding to the minimum critical stress.

the precise value of effective axial support stiffness k_x in the meridional membrane conditions

$$n_x = k_x u \quad (23)$$

With the equivalent column critical load also highly dependent on the buckling wave number n_w , would not the correct modelling of the meridional in-plane boundary condition be of crucial importance in determining the Authors' lower bounds?

80. In § 49 the Authors suggest that the buckling mode is set when the shell initially starts deforming with clamped end supports, and that with the loss of flexural rotational restraint through stiffener yielding near the supports, the shell would continue to deform in this mode. This then implies that the degeneration of stiffness at the ends is a result of the periodic flexural stresses associated with the buckling deformations of the clamped shell. If so, would not this stiffness degeneration occur only on those meridians undergoing inward radial deformations, i.e. those meridians on which the periodic flexural stresses throw additional compression on to the stiffener tips near the supports? In addition, have the Authors taken into consideration, when calculating this end mechanism load, the effect of axisymmetric prebuckling flexural deformations due to Poisson bulging? This would induce appreciable tensile stresses at the tips of the stiffener outstands near the end supports, and thereby strongly affect the load level at which this stiffener compressive yielding would occur. In any case if the buckling loads for the clamped shell are as high as those in Table 4, why at such relatively low load levels does the clamped shell seem to have developed such significant buckling deformations?

81. With the shell behaviour so critically dependent on stiffness k_x in the in-plane meridional boundary condition of equation (23), it might be expected that the buckling behaviour would be substantially influenced by the eccentricity of support in the test shells, once the Authors' postulated reduction to simple end supports occurs. Although not explicitly stated by the Authors the plastic mechanism that is assumed to form in the stiffener produces a hinged condition at the skin-stiffener junction.²⁸ Table 8 summarizes the results of a classical analysis in which the SS3 and SS4 boundary conditions are applied at the centroid of the skin and not at the orthotropic centroid, as assumed by the Authors and considered in Table 7. In Table 8 the analysis for which $Z_p = 0$ relates to the axial load applied at the orthotropic centroid, and for $Z_p = -1.0$ corresponds with the axial loading applied through the skin. Comparison of these results with those in Table 7 shows that the effects of both kinematic and static eccentricity in the loading of the shell play a crucial role in determining the critical load response. If the plastic mechanism has formed at the stiffener ends before buckling, do the Authors consider that such eccentricity of support and loading might also play an important part in their postulated buckling model?

82. With our own results indicating considerably greater potential imperfection sensitivity for stringer stiffened shells²⁹ than those of the Authors, and with these imperfection sensitivities consistent with those known to occur in the elastic buckling of isotropic shells (in contrast with those of the Authors) it would seem that this difference should be resolved. Would not a more cautious recommendation than that given in § 67 await the results of the further tests? At least some tests will need to be carried out on shells having greater control of

the all-important end boundary constraints and for which the effects of elastic non-linearity can be independently checked from the complicating effects of plasticity.

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Dr Ellinas and Dr Croll suggest that the lower bound prediction of the critical load of the shell should be based on an approach which neglects a certain non-linear membrane energy term of the shell. In our opinion such an approach has some merit in the prediction of buckling loads of a cylinder with significant (although as yet unquantified) level of initial imperfection which fail exclusively by elastic buckling. The modal interaction which must be considered in the prediction of shell behaviour in the vicinity of the critical load and in the postcritical state does not appear to be important in the behaviour of a clamped stiffened shell at a load well below the critical value and it is in this range that plastic yielding intervenes to initiate the failure. In the tests reported in the Paper, there was no noticeable non-linearity of the load-end shortening characteristic before plastic yielding (with the stiffness of the shell in good agreement with the E for the material); further, the number of waves of buckling $n_w/2$ observed in the tests is in closer agreement with the prediction of the classical theory than that of Dr Ellinas and Dr Croll. Hence the retention of the n_w predicted by the classical theory appears justified on the basis of experimental evidence at least. Our approach is based on the concept of neglecting all membrane energy contributions in the determination of the neutral equilibrium load with the value of n_w given by the classical theory. As mentioned in the Paper, a comparison of the neutral equilibrium load and classical critical load shows that the stringer-stiffened shells are far less imperfection sensitive than the elastic unstiffened shells—an observation in agreement with the findings of Singer and Rosen.³⁴ Our simplified approach should not be extended to unstiffened shells; it is intended only for shells with the levels of stringer-stiffening typical of offshore components, such as the ones tested in the investigation.

84. Regarding the implicit use of SS3 boundary conditions in the second stage of the analysis, the value of n_w determined for the initial clamped condition holds good for the second stage of the analysis as well and thus the use of SS3 conditions does not dictate the value of n_w . With the formation of plastic mechanisms at the ends of stiffeners, there is doubt as to what set of in-plane boundary conditions exists at the ends of the cylinder. It is evident (see, e.g., Table 7) that the use of SS3 conditions is conservative for the determination of the corresponding critical load and the stiffness of the springs of the column model makes the calculations simpler and is logically consistent with the use of the column model.

85. With regard to consideration of loss of stiffness of only the inward buckling regions of the shell in the simplified analysis, experimental observation has indicated that the formation one or two inward buckles signals the collapse (the limit point) of the shell. For a sufficiently accurate prediction of the collapse strength of the shell, it is necessary to consider only the most critical buckle.

86. We feel that it is not wise to rely on the consideration of tension induced at the stiffener tips as a result of bending induced due to restraints to Poisson expansion at the ends, because the bending induced is often extremely localized and changes sign at a short distance from the ends. We do not understand what Dr Ellinas and Dr Croll mean by 'significant buckling deformations' at 'rela-

tively low loads.' In the Paper we did not report the calculated or experimentally obtained out of plane deformations.

87. Our implicit assumption of $Z_p = 0$ (the load is applied through the centroid of the column model) results in a smaller value of n_w as shown in Table 8, which means a smaller stiffness of the springs of the column model, leading to a conservative estimate of collapse load. However, in the calculation of the performance of the tested specimens, n_w is that applicable for the initially clamped condition for which the value of Z_p is immaterial.

88. The method of approach proposed is thus seen to be based on a number of simplifying approximations, all of which appear to err on the safe side. In view of the uncertainties of practical shell behaviour, the assumptions are justified and the method can be used confidently in the preliminary design, once the worst imperfection in the Euler mode of any stiffened panel has been estimated.

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