Integral solutions of the gradually varied equation for rectangular and triangular channels

A. KUMAR

Dr K. J. Enever, The City University, and **Dr J. Allen**, Queen Mary College, University of London

We are interested in the Author's addition to the formulae available in this field having previously made a contribution ourselves.¹³ As the Author points out, his equation (12) reduces to the Bresse integral for very wide channels. We produced an approximate solution for very narrow channels of the form

$$x = \frac{h_0}{2i} \left[2 \frac{h}{h_0} - \ln \left| \frac{h+h_0}{h-h_0} \right| - \left(\frac{h_c}{h_0} \right)^3 \ln \left| \frac{h_0^2 - h^2}{h^2} \right| \right] + \text{constant} \quad (24a)$$

where h_c is the critical depth.

31. Putting this into the Author's non-dimensional terms it becomes

$$iX = Y - \frac{1}{2} \ln \left| \frac{Y+1}{Y-1} \right| - \frac{K}{2r_0} \ln \left| \frac{1}{Y^2} - 1 \right| + \text{constant} \quad (24b)$$

As the channel is narrow, r_0 is large and this can be simplified to

$$iX = Y - \frac{1}{2} \ln \left| \frac{Y+1}{Y-1} \right| + \text{constant} \quad (24c)$$

Taking equation (12) and putting $r_0 \rightarrow \infty$ gives

$$\begin{aligned} iX &= Y - \frac{1}{4} \ln \frac{Y(Y+1)}{(Y-1)^2} - \frac{1}{4} \ln \frac{(Y+1)}{Y} + \text{constant} \\ &= Y - \frac{1}{4} \ln Y - \frac{1}{4} \ln (Y+1) + \frac{1}{2} \ln (Y-1) - \frac{1}{4} \ln (Y+1) + \frac{1}{4} \ln Y + \text{constant} \\ &= Y - \frac{1}{2} \ln \left| \frac{Y+1}{Y-1} \right| + \text{constant} \quad (24c) \end{aligned}$$

Hence not only does equation (12) approach Bresse's solution as $r \rightarrow 0$, it also approaches our solution for $r \rightarrow \infty$.

32. We also produced a solution for a triangular channel of the form

$$\begin{aligned} x = \frac{h_0}{20i} & \left[20 \frac{h}{h_0} + \left[1 - \left(\frac{h_c}{h_0} \right)^5 \right] \left\{ 4 \ln |h-h_0| - (\sqrt{5}+1) \ln [2h^2 + (\sqrt{5}+1)hh_0 + 2h_0^2] \right. \right. \\ & \quad + (\sqrt{5}-1) \ln [2h^2 - (\sqrt{5}-1)hh_0 + 2h_0^2] \\ & \quad + 2[2(5-\sqrt{5})]^{1/2} \tan^{-1} \frac{h_0[2(5-\sqrt{5})]^{1/2}}{4h + (\sqrt{5}+1)h_0} \\ & \quad \left. \left. + 2[2(5+\sqrt{5})]^{1/2} \tan^{-1} \frac{h_0[2(5+\sqrt{5})]^{1/2}}{4h - (\sqrt{5}-1)h_0} \right\} \right] + \text{constant} \quad (25) \end{aligned}$$

DISCUSSION

33. If $(Y^5 - 1)$ from equation (18) is substituted into equation (17), and G_5 from equation (17) is substituted into equation (16), it is found that equations (16) and (25) are identical.

34. We also produced analytical solutions for horizontal channels and for channels of adverse bed slope and tested our solutions against both direct integration and experimental results obtained in the laboratory.

35. Gill¹⁴ shows how to obtain solutions for a very wide range of channel flows.

Mr Kumar

I am grateful to Dr Enever and Dr Allen for showing that the general solution (equation (12)) may be reduced to the simpler expression of equation (24c) for very narrow rectangular channels.

37. The solution in equation (24b) for gradually varied flow profiles in narrow rectangular channels¹³ may be derived from equation (8) by assuming that $(1 + r_0) \simeq r_0$ and that $(1 + r_0 Y) \simeq r_0 Y$. Equation (24c) may also be obtained from equation (8) by making an additional assumption, i.e. $K/r_0 = 0$.

38. Equation (24b) may be obtained from the general solution (equations (12) and (13)) by assuming

(a) that the multiplication factor for the function G_3 may be reduced to $(r_0 - K)6/4r_0$

(b) $G_3 = \ln [(Y^2 + Y)/(Y - 1)^2]/6$

(c) $\sqrt{(r_0 - 3)}$ and $\sqrt{(r_0 + 1)}$ are each equal to $\sqrt{r_0}$

(d) that the multiplication factor for the function G_4 may be reduced to $(-3K - r_0)\sqrt{3}/2r_0$

(e) $G_4 = \ln [(Y + 1)/Y]/2\sqrt{3}$.

39. The applicability of equations (24b) and (24c) should be considered in view of these assumptions.

40. It should be noted that for rectangular channels

$$\left(\frac{h_c}{h_0}\right)^3 = Y_c^3 = \frac{K}{1 + r_0}$$

and that for narrow channels

$$Y_c^3 = \frac{K}{r_0}$$

h_c is the critical depth for the discharge. For mildly sloping channels $Y_c < 1$, for channels with critical bed slopes $Y_c = 1$ and for channels with steep slopes $Y_c > 1$.

41. Gradually varied flow profiles in mildly sloping ($K/r_0 = 0.01$ and 0.1), critically sloping ($K/r_0 \simeq 1$) and steeply sloping ($K/r_0 = 2$) narrow rectangular channels with $r_0 = 50$ and 100 were calculated by using the general solution (equations (12) and (13)) and equations (24b) and (24c). The calculated non-dimensional profile lengths iL between specific profile end depths for various types of surface profile in a channel with $r_0 = 50$ are given in Table 3.

42. Percentage differences in profile lengths D_1 calculated from the expression

$$D_1 = \frac{iL(\text{equation (24a)}) - iL(\text{equation (12)})}{iL(\text{equation (12)})} \times 100$$

are also given in Table 3.

43. Equation (24c) gives grossly inaccurate profile lengths for small values of Y , i.e. in the M3, C3 and S3 profiles. Profile lengths calculated from equation (24c) for larger values of Y in the M2 and S2 profiles and for $Y > 1$ in the C1 and S1 profiles are also inaccurate. Equation (24c) gives satisfactory results for M1 profiles in very narrow ($r_0 > 100$) channels with very mild slopes ($K/r_0 < 0.01$). It is thus of very limited use in practice.

44. Equation (24b) was generally also found inadequate for the computation of

Table 3. Profile lengths in narrow channels, $r_0 = 50$

$\frac{K}{r_0}$	Profile depths	Type of profile	iL			$D_1, \%$
			Equation (24c)	Equation (24b)	Equation (12)	
0.01	0.001–0.2	Part of M3	0.0027	0.0504	0.5567	–90.9
0.10	0.001–0.2		0.0027	0.5291	0.7693	–31.2
0.01	0.5–0.99	Part of M2	1.6073	1.5823	1.4706	7.6
0.10	0.5–0.99		1.6073	1.3575	1.2533	8.3
0.01	1.3–4.0	Part of M1	3.4629	3.4588	3.5315	–2.0
0.10	1.3–4.0		3.4629	3.4214	3.4950	–2.1
1.0	0.001–0.99	C3	1.6539	7.1995	4.064	77.1
2.0	0.001–0.99	S3	1.6539	16.055	10.652	50.7
2.0	1.001–1.2	Part of S2	2.8007	2.2296	2.0906	6.6
1.0	1.3–4.0	Part of C1	3.4629	3.0474	3.1299	–2.6
2.0	1.3–4.0	Part of S1	3.4629	2.6318	2.7241	–3.3

profiles in which $Y < 1$. The error (D_1 in Table 3) increases with decreasing values of profile depth Y .

45. Equation (24b) was found to be grossly inaccurate, even in channels with $r_0 = 100$, for the M3, C3 and S3 profiles in which Y is small. The error is greater in narrower channels with milder slopes for mildly sloping channels and smaller in narrower, steeply sloping channels.

46. Profiles M2 and S2, in which the profile depths Y are nearer 1, are also inaccurately described by equation (24b).

47. Equation (24b) gave nearly (97–98%) correct profile lengths for the M1, C1 and S1 profiles in which $Y > 1$.

48. The general solution may also be used to assess the applicability of the Bresse solution for wide, shallow channels. Equations (12) and (13) and equations (6) and (7) were used to calculate the profile lengths of various types of surface profile in wide channels. Profile lengths in channels with $r_0 = 1/50$ and the differences D_2 , calculated from the expression

$$D_2 = \frac{iL(\text{equation (6)}) - iL(\text{equation (12)})}{iL(\text{equation (12)})} \times 100$$

are given in Table 4.

Table 4. Profile lengths in wide channels, $r_0 = 1/50$

K	Profile depths	Type of profile	iL		$D_2, \%$
			Equation (6)	Equation (12)	
0.1	0.001–0.2	Part of M3	0.0195	0.0194	0.25
0.1	0.5–0.99	Part of M2	0.8602	0.8693	–1.04
1.0	0.001–0.99	C3	0.9890	0.9610	2.91
2.0	0.001–0.99	S3	3.0050	2.9566	1.63
2.0	1.001–1.25	S2	1.5149	1.4549	4.12
0.1	1.3–4.0	Part of M1	3.0073	3.0150	–0.25
1.0	1.3–4.0	Part of C1	2.9990	3.0490	–1.63
2.0	1.3–4.0	Part of S1	2.3584	2.3766	–0.93

DISCUSSION

49. The Bresse equation gives nearly correct profile lengths for the three types of surface profile in mild sloping channels ($K < 1$) and for the C1 and S1 profiles in which $Y > 1$; it was less accurate for profile types 2 and 3 in critical and steeply sloping channels. Hence, Bresse's solution may be used to compute sufficiently accurate profiles in mildly sloping channels with $r_0 \leq 1/50$ and in critical and steeply sloping channels with $r_0 < 1/50$.

50. The applicability of equation (24b) and the Bresse solution (equation (6)) to channels in which the normal depths of flow are not very different from widths was also considered by computing surface profiles in channels with $r_0 = 2$ and 5. The non-dimensional profile lengths calculated from these equations and those obtained from the general solution are given in Tables 5 and 6; percentage differences D_1 and D_2 in profile lengths are also shown.

51. The results of the calculations show that equation (24b) is not suitable for the calculation of type 3 ($Y < 1$) and type 2 ($Y \approx 1$) profiles; it provides approximately correct (106.6%, 97% and 95.3%) answers for type 1 profiles ($Y > 1$).

52. The Bresse solution provided the least inaccurate (84%) profile lengths for the M1 profiles; it is generally unsuitable for the calculation of surface profiles in rectangular channels in which the calculated normal depths of flow are greater than the widths.

53. It is therefore advisable to use the general solution (equations (12) and (13)) for the computation of surface profiles in narrow and narrow and deep rectangular channels.

54. The function G_5 may also be derived by using the general solution¹² for binomial equations ($Y^n - 1 = 0$, where n is an integer).

55. The solutions given by Gill¹⁴ are based on a simplified version of the gradually

Table 5. Profile lengths in channels, $r_0 = 2$

$\frac{K}{r_0}$	Profile depths	Type of profile	iL			$D_1, \%$	$D_2, \%$
			Equation (6)	Equation (24b)	Equation (12)		
0.1	0.001-0.4	Part of M3	0.0745	0.5842	0.0475	1128	56.7
0.1	0.5-0.99	Part of M2	0.7102	1.3575	1.2049	12.6	-41.0
1.5	0.001-0.99	Part of C3	0.2146	11.627	1.7542	562	-87.7
2.0	0.001-0.99	Part of S3	7.0373	16.0557	2.7957	474	151
0.1	1.01-4.0	Part of M1	4.1001	5.1931	4.8699	6.63	-15.8
1.5	1.01-4.0	Part of C1	0.2146	2.4894	3.2782	-24.0	-93.4
2.0	1.01-4.0	Part of S1	1.1731	1.5238	2.7098	-43.7	-56.7

Table 6. Profile lengths in channels, $r_0 = 5$

$\frac{K}{r_0}$	Profile depths	Type of profile	iL			$D_1, \%$	$D_2, \%$
			Equation (6)	Equation (24b)	Equation (12)		
0.01	0.001-0.2	Part of M3	0.0059	0.0504	0.0079	536	-24.4
0.01	0.5-0.99	Part of M2	0.9351	1.5823	1.4424	9.7	-35.1
1.2	0.001-0.99	Part of C3	11.069	8.9706	2.3184	286	377
2.0	0.001-0.99	Part of S3	19.133	16.055	4.8623	230	293
0.01	1.01-4.0	Part of M1	4.3083	5.3669	5.1585	4.03	-16.4
1.2	1.2-4.0	Part of C1	0.5586	3.0707	3.1733	-3.23	-82.3
2.0	1.2-4.0	Part of S1	1.2345	2.6223	2.7512	-4.68	-55.1

varied flow equation (equation (4)). Following Bakhmeteff⁷ and Chow,¹ the profile slope equation may be modified and stated as

$$\frac{dY}{dX} = i \frac{1 - Y^{-N}}{1 - Y_c^M Y^{-N}} \quad (26)$$

The exponents N and M , for a specific channel, are obtained by relating $Q/\sqrt{i}$ to h and Q to h in successive ranges of profile depth h .

56. The integration of the profile is then reduced to the integration of an inverse binomial function such as $(Y^p - 1)^{-1}$, where $p = N$, $N - M + 1$. Bakhmeteff⁷ and Chow¹ gave numerical values of the function for various values of Y and p to facilitate the computation of surface profiles.

57. Gill¹⁴ obtained closed form integration of the binomial function $(Y^p - 1)^{-1}$ for various values of the exponent p enabling direct integration of the simplified differential equation (26). The exact equations for wide rectangular, wide parabolic and triangular channels contain binomial expressions with $p = 3, 4$ and 5 , respectively.

58. Dr Enever and Dr Allen have provided additional relevant information. Certain aspects of gradually varied flow and some specific flow profiles need to be investigated further; also, it may be useful in some cases to seek alternative solutions for profiles in terms of directly applicable non-dimensional parameters.

References

13. ALLEN J. and ENEVER K. J. Water surface profiles in gradually varied open-channel flow. *Proc. Instn Civ. Engrs*, 1968, **41**, Dec., 783-811.
14. GILL M. A. Exact solution of gradually varied flow equation. *Proc. Am. Soc. Civ. Engrs*, 1976, **102**, Sept., 1353-1364.