Two additional methods of direct solution of the Colebrook-White function

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The Author is to be commended for presenting an explicit approximation (equation (5)) to the Colebrook-White function. The percentage error values shown in Table 1 indicate that equation (5) gives a better estimate of the friction factor than other explicit approximations.^{2,3}

15. The development of equations (8)-(11) and the modifications to the Moody diagram are superfluous as most practising engineers now have access to programmable computing capacity for solution of simple pipe problems.¹² Alternatively the Wallingford charts of Ackers¹⁰ have proved adequate.

16. The main value of equation (5) is in its potential application to the analysis of steady flow in pipe networks, in particular for loop methods. For the current value of discharge Q , loop methods require the head loss h and the derivative dh/dQ . The rigorous Colebrook-White equation (equation (1)) has been used by Barlow and Markland¹³ but this carries a penalty in computing time as the friction factor λ must be found iteratively. The approximation to λ due to Wood⁹ has been used¹⁴ to avoid this iteration. Equation (5) may be used to obtain a more accurate solution, as now outlined.

17. Following the development of Barlow and Markland,¹³ take the expression for head loss h as

$$h = \lambda \frac{8l}{g\pi d^5} Q^n = \lambda K Q^n \quad \dots \dots \dots (19)$$

λ is obtained from equation (5). The exponent n is found by using

$$n = 2 + \frac{R}{\lambda} \frac{d\lambda}{dR}$$

in conjunction with equation (5). The result is

$$n = 2 - \frac{7.93 \sqrt{\lambda}}{\frac{k}{3.7d} R^{0.89} + 5.13} \quad \dots \dots \dots (20)$$

The required derivative is then

$$\frac{dh}{dQ} = \frac{nh}{Q} \quad \dots \dots \dots (21)$$

18. These expressions may be incorporated in an efficient computer program sub-routine. For each pipe the three physical data values required are d , $k/3.7d$ and K .

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The following alternative procedure is based on the fact that generally one of the two terms of the Colebrook-White formula is more important than the other. For example, Ackers¹⁰ showed that for fluids of small viscosity, the velocity given by the rough pipe law should be reduced by $8\nu/k$. This idea can be extended to a wider range of viscosities by replacing the figure 8 by a variable coefficient C_r , which depends on Reynolds number and the relative roughness. Thus

$$v = v_{\text{rough}} - C_r \nu / k \quad (22)$$

20. Alternatively, when the viscous term is the more important one a similar treatment gives

$$v = v_{\text{smooth}} - C_s d s k / \nu \quad (23)$$

Fig. 4

21. Values of C_r and C_s are shown in Fig. 4. The thick line represents the case where both terms of the Colebrook-White formula carry equal weight. For a pipe of known diameter, a comparison of the two terms in equation (6) shows that the rough term governs when

$$\sqrt{(sd)} > 2.1 \nu/k \quad \dots \dots \dots (24)$$

and for a pipe of unknown diameter, where the discharge and slope are known, the corresponding relation can be obtained from equation (9)

$$Q^{0.19} s^{0.4} > 3.4 \nu/k \quad \dots \dots \dots (25)$$

Fig. 5

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22. Solutions of the rough and smooth pipe laws can be simplified by making the powers of d the same inside and outside the logarithm. The following expressions are obtained for the rough law and smooth law respectively

$$Q/\sqrt{(sk^5)} = 0.105 (x \log_{10} x) \quad \text{where } x = (3.71 d/k)^{2.5} \quad (26)$$

$$Qs^{1/3}/\nu^{5/3} = 1.63 (x \log_{10} x) \quad \text{where } x = (1.758 \sqrt{(sd^3)/\nu})^{1.67} \quad (27)$$

In the latter case x is related to Reynolds number and equations (26) and (27) are given in Fig. 5.

23. If the pipe diameter is known, the problem is simple. If it is unknown, it may be obtained approximately by assuming the rough or smooth law to apply, as indicated by equation (25). Then the correction term in either equation (22) or (23), whichever is appropriate, is multiplied by $\pi d^2/4$ to give δQ . The true diameter is given by the formula

$$\frac{\text{true } d}{\text{approximate } d} = \left(\frac{Q + \delta Q}{Q} \right)^{0.38} \quad (28)$$

Mr A. K. Jain, Department of Civil Engineering, University of Roorkee, on deputation from Government College of Engineering and Technology, Raipur

The Author has proposed useful explicit equations yielding direct solution for the friction factor, pipe diameter and the discharge. Such equations should prove extremely useful to those dealing with the analysis of pipe flow.

25. I have been associated with a similar study which yielded almost identical equations, except the one for the pipe diameter which in my case is exponential. An explicit equivalent of the Colebrook-White equation (1) was obtained by suitably combining the explicit equations for turbulent flow with smooth and rough boundaries. The explicit equation for turbulent flow with a smooth boundary was obtained using curve fitting technique, and is

$$1/\sqrt{\lambda} = 1.8 \log R - 1.5146 \quad (29)$$

the similar equation for turbulent flow in a rough pipe being

$$1/\sqrt{\lambda} = 1.14 - 2 \log (k/d) \quad (30)$$

The resulting explicit equation¹⁵ for the transition region, which is a mathematical approximation of equation (1), is

$$\frac{1}{\sqrt{\lambda}} = 1.14 - 2 \log \left(\frac{k}{d} + \frac{21.25}{R^{0.9}} \right) \quad (31)$$

which can also be written as

$$\frac{1}{\sqrt{\lambda}} = -2 \log \left(\frac{k}{3.7d} + \frac{5.74}{R^{0.8}} \right) \quad (32)$$

26. Equation (32) computes the friction factor to within $\pm 1.0\%$ of that given by equation (1). Comparison of my equation with equation (5) shows that the equations are practically the same; the constants and the exponents associated with the Reynolds number are only slightly different.

27. The conventional method of solving pipe flow problems, namely determination of diameter, discharge and the head loss, involves a trial procedure. There are, however, graphical solutions proposed by many researchers aimed at direct solution of these problems. I have been associated with a study aimed at finding explicit relationships for these pipe flow problems.¹⁶ The resulting equations are presented below.

28. For pipe diameters, in the particular cases of turbulent flow in smooth and rough pipes respectively, the following equations were obtained

$$d_* = 0.66 \nu_*^{0.04} \quad (33)$$

$$d_* = 0.66 k_*^{0.05} \quad (34)$$

in which

$$d_* = d(g_s/Q^2)^{0.2}$$

$$\nu_* = (1/g_s Q^3)^{0.2}$$

$$k_* = k(g_s/Q^2)^{0.2}$$

29. For the transition region the following equation has been fitted

$$d_* = 0.66 (k_*^{1.25} + \nu_*)^{0.04} \quad (35)$$

This reverts to equations (33) and (34) for smooth and rough turbulent flow cases. It computes the diameter within $\pm 2\%$ of the value obtained using the Colebrook-White equation.

30. For discharge, the Colebrook-White equation (1), by suitable combination of variables, can be expressed as

$$\frac{Q}{d^2 \sqrt{(gsd)}} = -\frac{\pi}{\sqrt{2}} \log \left(\frac{k}{3.7d} + \frac{1.78}{d \sqrt{(gsd)}} \right) \quad (36)$$

which corresponds to equation (6) with a slight difference in the constant associated with ν . Equation (36) can be written for velocity as

$$\frac{V}{\sqrt{(gsd)}} = -2\sqrt{2} \log \left(\frac{k}{3.7d} + \frac{1.78}{d \sqrt{(gsd)}} \right) \quad (37)$$

Equations (36) and (37) yield the closed form solution of equation (1) for discharge and velocity respectively.

31. The Darcy-Weisbach equation for head loss and equation (32) give the head loss in terms of the known variables as

$$h = \frac{0.203 l Q^2}{gd^5 \left[\log \left(\frac{k}{3.7d} + \frac{5.74}{R^{0.9}} \right) \right]^2} \quad (38)$$

Equation (38) is identical in form with equation (7) except for having a different constant and exponent of the pipe Reynolds number.

32. Equation (9) computes the diameter to $\pm 1\%$ accuracy as compared with equation (35) which determines the diameter to within $\pm 2\%$ of the value obtained from the use of the Colebrook-White equation (1). Equation (32) for the friction factor gives more accurate results than equation (5) (an accuracy of $\pm 1.0\%$ as compared with $\pm 2.5\%$). The same trend is reflected in equations (38) and (7) for the piezometric slope $s = h/l$.

33. I wish to thank Dr K. G. Ranga Raju, Professor of Civil Engineering, University of Roorkee, for his help in the preparation of this discussion.

Professor Barr

Mr Feitz rightly emphasizes the importance of the determination of unknown piezometric gradients in network calculations. He dismisses the need for their work on the other problems and on graphical solutions. I believe that the pursuit of the ideal will always be a challenge. The capacity to solve any of the basic problems with a single diagram and in dimensional terms remains the ultimate aim in this area. However, accepting the priority rating adopted by Mr Feitz, I give in my reply to Mr Jain, some more accurate numerical approximations for friction factor and hence unknown gradient. While I am conscious of the effect of the law of diminishing returns—it is easy to solve by iteration on the simplest programmable computing system—I nevertheless hope that the advantage in reducing computing time in large network calculations may again be found by using these improved approximations.

35. It is interesting that Mr Jain's work should have so closely paralleled that described in the Note. The form of equation (5) and equations (31) and (32) was published in 1972.⁷ Nevertheless, the step necessary from basic approximations to the smooth turbulent law, as given many years previously, is a small one. It is surprising that this

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form of approximate transition function was not used earlier. Nikuradse¹⁷ in 1932 gave

$$1/\sqrt{\lambda} = 1.8 \log (R/7) \quad (39)$$

which can be re-arranged as

$$1/\sqrt{\lambda} = 2 \log (R^{0.9}/5.762) \quad (40)$$

whereas equation (29) gives

$$1/\sqrt{\lambda} = 2 \log (R^{0.9}/5.719) \quad (41)$$

Colebrook¹ quoted equation (39) and White⁸ suggested a slightly different version (equation (4)).

36. I am not sure of the basis of Mr Jain's claim that equations (31) and (32) give a computation of friction factor to within $\pm 1\%$, whereas equation (5) has an accuracy of $\pm 2.5\%$. Using equation (32) and with a Reynolds number of 3000, calculations for d/k values from 10 by 10 times to 10^7 give errors of $+3.25\%$, $+3.58\%$, $+2.47\%$, $+2.25\%$, $+2.23\%$, $+2.23\%$ and $+2.23\%$ respectively. Thus, at this limiting condition, his compromise gives less accurate results than does equation (5). However, his compromise is, by and large, better at higher values of Reynolds number.

37. Seeking to elucidate the matter further, I found there are two distinct reasons for inaccuracy in this case of determination of friction factor λ , to which, in degree of accuracy, the case of evaluation of piezometric slope corresponds. First, the smooth turbulent element of the whole transition approximation may be inaccurate. Second, it is not the smooth turbulent resistance for a particular flow and pipe size which is used in the Colebrook-White transition, but an element for which the shear stress corresponds to the transition shear stress. Thus even a perfect explicit assessment of smooth turbulent resistance will fail to provide a completely accurate assessment of Colebrook-White transition resistance over the whole transition region.

38. Without undue complication, a correcting factor for this second feature can be devised, which gives quite good results over the whole transition region. An evaluation of this factor gave $[1 + R^{0.52}/29(d/k)^{0.695}]$ as the divisor of the smooth law element of the approximation for friction factor, as it appears inverted in the approximate transition function.

39. As regards the basic actual smooth turbulent approximation, the effect of errors so engendered may sometimes compensate for the errors which the above factor corrects with an exact smooth turbulent evaluation. At other points the two forms of error are cumulative, which is the reason for the largest errors shown using Mr Jain's compromise, and the only slightly smaller errors occurring with equation (5). Something close to Nikuradse's explicit approximation appeared in an equation given by Techo *et al.*¹⁸

$$\lambda = [0.86859 \ln (R/(1.964 \ln R - 3.8215))]^{-2} \quad (42)$$

This arises almost exactly from the writing of a smooth turbulent equation as a first iteration from the Nikuradse approximation, equation (39).

$$1/\sqrt{\lambda_{\text{smooth}}} = 2 \log \{R/[2.51 \times 1.8 \log (R/7)]\} \quad (43)$$

So an even more accurate approximation can be obtained by writing a second iteration equation on this basis; in this case a slightly different basic approximation is introduced, as was used in Tables 2 and 3

$$1/\sqrt{\lambda_{\text{smooth}}} = 2 \log \{R/[5.02 \log (R/5.02 \log (R^{0.89}/5.2016))]\} \quad (44)$$

Then for either a single or two primary iterations very reasonable direct transition solutions are given by use of equation (45) which includes the correction factor

$$\frac{1}{\sqrt{\lambda}} = 2 \log \left\{ \frac{1}{3.7d/k} + \frac{2.51}{R\sqrt{\lambda_{\text{smooth}}} (1 + R^{0.52}/29(d/k)^{0.695})} \right\} \quad (45)$$

If the immediate accuracy is still insufficient, a first full iteration of the transition function gives what may probably be regarded as a sufficiently accurate solution for any practical

Table 2. Percentage error in assessment of friction factor for Reynolds numbers from 3×10^3 to 3×10^7 and d/k values from 10 to 10^7 , using approximate transition function based on single iteration of smooth law approximation with correction factor (upper figures), with error after one whole function iteration also shown (lower figures)

	3×10^3	3×10^4	3×10^5	3×10^6	3×10^7
10	+0.36 -0.01	-0.09 0.00	-0.03 0.00	-0.01 0.00	0.00
10^2	-0.15 +0.02	+0.02 0.00	-0.10 0.00	-0.03 0.00	-0.01 0.00
10^3	-0.53 +0.09	-0.12 +0.01	-0.02 0.00	-0.08 0.00	-0.03 0.00
10^4	-0.27 +0.06	-0.12 +0.02	-0.11 +0.01	-0.01 0.00	-0.05 0.00
10^5	-0.30 +0.06	-0.01 0.00	-0.09 +0.01	-0.12 +0.01	+0.01 0.00
10^6	-0.29 +0.05	+0.03 -0.01	-0.02 0.00	-0.13 +0.01	-0.14 +0.01
10^7	-0.28 +0.05	+0.04 -0.01	0.00	-0.09 +0.01	-0.17 +0.01

Table 3. Percentage error in assessment of friction factor for Reynolds numbers from 3×10^3 to 3×10^7 and d/k values from 10 to 10^7 , using approximate transition function based on double iteration of smooth law approximation with correction factor (upper figures), with error after one whole function iteration also shown (lower figures)

	3×10^3	3×10^4	3×10^5	3×10^6	3×10^7
10	+0.41 -0.01	-0.09 0.00	-0.03 0.00	-0.01 0.00	0.00
10^2	+0.06 -0.01	+0.01 0.00	-0.01 0.00	-0.03 0.00	-0.01 0.00
10^3	-0.23 +0.04	-0.16 +0.02	-0.03 0.00	-0.08 0.00	-0.03 0.00
10^4	-0.05 +0.01	-0.17 +0.02	-0.11 +0.01	+0.01 0.00	-0.05 0.00
10^5	+0.03 -0.01	-0.06 +0.01	-0.09 +0.01	-0.05 0.00	+0.01 0.00
10^6	+0.05 -0.01	-0.02 0.00	-0.03 +0.01	-0.05 0.00	-0.01 0.00
10^7	+0.05 -0.01	-0.01 0.00	-0.01 0.00	-0.01 0.00	-0.02 0.00

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Table 4. Percentage error in assessment of friction factor for Reynolds numbers from 3×10^3 to 3×10^7 and d/k values from 10 to 10^7 , using approximate transition function based on amplified smooth law approximation with correction factor (upper figures), with error after one whole function iteration also shown (lower figures)

	3×10^3	3×10^4	3×10^5	3×10^6	3×10^7
10	+0.42 -0.01	-0.07 0.00	-0.03 0.00	-0.01 0.00	0.00
10^2	+0.07 -0.01	-0.02 0.00	-0.10 0.00	-0.03 0.00	-0.01 0.00
10^3	-0.22 +0.04	-0.26 +0.03	0.00	-0.07 0.00	-0.03 0.00
10^4	-0.04 +0.01	-0.32 +0.04	-0.04 0.00	+0.07 0.00	-0.04 0.00
10^5	+0.04 -0.01	-0.21 +0.03	0.00	+0.11 -0.01	+0.08 0.00
10^6	+0.05 -0.01	-0.17 +0.02	+0.07 -0.01	+0.16 -0.02	+0.07 -0.01
10^7	+0.06 -0.01	-0.16 +0.02	+0.09 -0.01	+0.20 -0.02	+0.08 -0.01

purpose. These results are shown in Tables 2 and 3 where the upper figures represent the direct transition solution and the lower figures the result with one full iteration using equation (1) for the second round calculation, and again introducing $1/\sqrt{\lambda}$ from the direct solution. In preparing these Tables, I have increased the coverage to Reynolds numbers of 3×10^3 by 10 times to 3×10^7 , all by d/k values of 10 by 10 times to 10^7 .

40. It has already been shown that slightly different smooth turbulent approximations are appropriate to different friction factors (and Reynolds number ranges). If the range can be restricted to, say, Reynolds number variation of 100 times, a good approximation can be calculated and used with the correction factor. Carrying this idea further, an overall approximation can be formed by controlling the index and coefficient to correspond successively to the desired small range values.

$$\frac{1}{\sqrt{\lambda_{\text{smooth}}}} = 2 \log \left\{ \frac{R^{0.858 + 0.0078 \log R}}{2.735 + 0.595 \log R} \right\} \quad (46)$$

41. This is a true smooth law approximation, rather than a first or second iteration combination. It would be pointless to place it in an approximate transition function without use of the correction factor. When this is done, i.e. equation (45) is used, the result is satisfactory and after a first whole function iteration is very good. Results are shown in Table 4.

42. Dr Sarginson provides an alternative semi-graphical method using a new correlation of departures from either smooth turbulent or rough turbulent conditions. This, in combination with re-arranged forms of the individual smooth and rough turbulent resistance laws, allows a fairly convenient solution procedure. Thus he is proposing a shift in emphasis towards coalesced mode working. I would agree with this, although my approach to this forward step differs from that outlined. Certainly there is increasing evidence that the assumption of a single transition route is insufficient for some practical problems. This being so, there is advantage in developing semi-graphical methods which avoid the use of a full resistance diagram.

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