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DISCUSSION

The ultimate strength of biaxially restrained columns

K. S. VIRDI & P. J. DOWLING

Dr S. Vinnakota, Swiss Federal Institute of Technology, Lausanne

The theory developed by the Authors and their computer program should be especially useful in making parametric studies of reinforced concrete columns and composite steel-concrete columns, for which torsion is effectively negligible, as assumed in the theory. Has the program already been used to check the validity of the existing design methods of reinforced concrete or composite steel columns?

49. The Authors have pointed out that for columns in antisymmetrical bending 16-20 segments have to be used to obtain acceptable results, compared with the 8 segments sufficient for single curvature bending. I came to a similar result for biaxially restrained bare steel, beam columns. So, to increase the accuracy of the finite difference method, I made use of two half segments at the two extremities of the beam column as shown in Fig. 9. Although the corresponding finite difference expressions are slightly more involved,¹⁹ the number of segments, the size of matrices and the computer time could be reduced considerably.

50. The sections selected by Gent and Milner¹⁷ are not truly representative of torsionally weak, rolled steel sections used in practice. Also, the loading condition—very heavy axial load at failure—used in their experiments may not include all loading types that are encountered in reality. In effect, for most of the columns forming part of rigid jointed frames, the major axis bending moments will be proportionally more important. For such columns, as the sections become plastic there is a continuous shift in the shear centre, and a shift and rotation of the principal axes of the remaining elastic core under the combined influence of major and minor axis moments, residual stresses and initial crookedness. So, theoretically torsion effects cannot be dissociated from

Fig. 9. Finite difference grid¹⁹

Fig. 10. Beam column¹⁹

bending effects. As the effect of torsion is to reduce the ultimate strength, I do not think that, as far as bare steel columns are considered, any general conclusions on the effect of torsion should be drawn, based on a single comparison made by the Authors.

51. There are more general and exact procedures¹⁹ for the study of the inelastic stability of restrained bare steel, beam columns. In this study, the theory is developed for a member of an arbitrary, thin-walled, open cross-section. The three non-linear, non-homogeneous, coupled differential equations of equilibrium obtained for a bar subjected to axial force, end moments and transverse loads (Fig. 10) loaded into the plastic range are given by

$$E\tilde{I}_{y0}u'' + E\tilde{I}_{xy0}v'' + E\tilde{I}_{x0}\theta'' - Nu + \left(1 - \frac{z}{L}\right)\alpha_{y0}u' - \frac{z}{L}\alpha_{y1}u' = \left(1 - \frac{z}{L}\right)M_{y0} + \frac{z}{L}M_{y1} - (X_0 - u_1)N + M_{qx} - f_y \quad (57)$$

$$E\tilde{I}_{xy0}u'' + E\tilde{I}_{x0}v'' + E\tilde{I}_{y0}\theta'' - Nv - \left(1 - \frac{z}{L}\right)\alpha_{x0}v' + \frac{z}{L}\alpha_{x1}v' = -\left(1 - \frac{z}{L}\right)M_{x0} - \frac{z}{L}M_{x1} - (Y_0 - v_1)N + M_{qy} + f_x \quad (58)$$

$$E\tilde{I}_{\omega x0}u'' + E\tilde{I}_{\omega y0}v'' + E\tilde{I}_{\omega0}\theta'' - (GK_T + \bar{K})\theta = -\left(1 - \frac{z}{L}\right)M_{\omega0} - \frac{z}{L}M_{\omega1} + \bar{K}\theta_1 - M_m - f_\theta \quad (59)$$

Fig. 11. Displacements of the pole O

where u , v and θ are the deformations (Fig. 11), at abscissa z , of the arbitrary point O of the cross-section taken as the pole of the sectorial areas, ωu_1 , v_1 and θ_1 are the geometrical imperfections of the bar at section z . The ordinates X , Y are measured with respect to a system of axes X - Y moving parallel to the arbitrary co-ordinate system x - y - z fixed in space. When the section enters plastic range, the rigidities $\tilde{I}_{x_0}$, $\tilde{I}_{xy_0}$ and so on are calculated for the remaining elastic core at section z . The quantities f_x , f_y and f_θ are functions of the part of the bending moments and bimoments resisted by the plasticized part of the cross-section. M_{qx} , M_{qy} and M_m represent the influence of the external transverse loads q_x and q_y . The influence of residual stresses appears in the functions of f_x , f_y and f_θ , indirectly in the calculation of the elastic core and in the calculation of the term $\bar{K} (= \int \sigma(x^2 + y^2) dA)$.

52. Although equations (57)–(59) appear involved, they have the advantage that a single computer program can be used to study the flexural torsional stability of beam columns, lateral buckling of beams, torsional buckling of columns and planar stability of beam columns. Several numerical examples using this program and earlier versions of it have been presented elsewhere.^{20–24}

Dr T. K. Sen, Department of Civil Engineering, Imperial College of Science and Technology

Notation

E	elastic modulus
I_y	moment of inertia about the Y - Y axis (weak axis)
K_{byb}	stiffness of minor axis restraining beam at the lower end
K_{byt}	stiffness of minor axis restraining beam at the top end
K_o	column rotational stiffness, $4EI_y/L$
P	axial load
r_y	radius of gyration about y axis
β	end moment ratio (+1.0 for uniform single curvature bending)

Fig. 12. Load-deflexion characteristics (restrained column)

Fig. 13. Variation of twist with major axis moment

Fig. 14

The Authors are to be complimented for developing a method for generalized end loading which is applicable to a wide variety of cross-sections. However, in the analysis, twisting effects have been ignored. The method may therefore result in significant errors when applied to H or I sections where torsional instability may be dominant. To illustrate the point consider a practical column in a building frame where major axis beams are designed according to plastic theory, and therefore offer no restraints to the column. The column is restrained about the minor axis only. Figs 12 and 13 show the load-deflexion response of such a restrained H column ($L/r_y = 74$) as obtained from a general non-linear biaxial program,²⁵ taking into account torsional effects. The load path followed in the analysis was constant axial load of 150 tons applied first followed by major axis end moment to failure. The column was subjected to a major axis moment gradient ($\beta = 0.15$). The major and minor axis deflexions and twist results presented are for mid-column height only.

54. Results for the restraining beam rotational stiffness of 12 000 ton in./radian ($K_{bvb}/K_c = 0.58$) and 20 000 ton in./radian ($K_{bvb}/K_c = 0.97$) are presented in Figs 12 and 13. A multi-linear stress-strain curve with no precise yield point (Fig. 14) was used in the computations. Minor axis imperfections were $L/1400$.

55. It may be seen from Figs 12 and 13 that minor axis deflexions are small but twist values are large and indicate that the column could fail by torsional buckling, whereas the Authors' method would indicate a flexural failure about the major axis. Designers should take note of the possibility of this type of torsional failure which should be investigated.

Dr A. K. Basu, Indian Institute of Technology

Of the methods available for the analysis of columns in the inelastic range the most common are the method of Newmark¹⁰ (which involves integration of curvatures for finding the deflexions) and the method of classical finite differences (where the curvatures are calculated from deflexions using finite difference approximations). As numerical differentiation is generally less accurate than numerical integration for the same size of mesh length and the same order of the interpolation polynomial, it may be surmised that the Authors' procedure, being based on the finite difference method, will need a larger number of divisions in the column length than the Newmark method for the same accuracy. The total amount of work needed for the calculation will depend on the number of iterations required to reach the peak load and also on other details of the algorithm. Have the Authors made any comparison to study the efficiency of the proposed method in relation to that of other possible methods?

57. There are a few aspects of the proposed method which do not seem to be entirely satisfactory. As the Authors themselves admit (§ 34), 'Most of the computer time is spent in generating the moment-thrust-curvature relations' for the column cross-section under biaxial bending because they used a method based on dividing the cross-section into a number of small areas and summing the contributions of these areas. The computational efficiency can be drastically improved by replacing this procedure by a closed form technique.

58. The Authors have used only a load increment algorithm, which indicates the attainment of the maximum load by the non-convergence of the iterations. It may involve reduction of the load increment a number of times to obtain the failure load to the desired degree of accuracy. Furthermore, it does not allow the descending branch of the load-deflexion curve to be obtained. In § 6 the Authors stated that the computation follows a possible ultimate load testing procedure so it would be more appropriate to switch over to a procedure of deflexion increment or strain increment as the failure load is approached as would normally be done during testing in the laboratory.

59. Perhaps the most serious reservation regarding the method is that it does not check the equilibrium between the external and the internal moment at the end sections. Thus, if material failure were to occur at or near the end sections the column would not

know. This would give rise to rather large unsafe errors in the case of columns with double curvature bending as the Authors have themselves found (§ 37). Moreover, the consideration of the elastic case would readily show that the end slope is related to the end curvature, and both would require to be expressed in terms of a fictitious point at each end. The end slope expressions used by the Authors (equations (36)–(39)) are not in any way related to the end curvatures and may therefore be quite inaccurate. For columns with large end restraints this may lead to large errors.

60. My colleagues and I have been involved with the development of analytical procedures for uniaxially and biaxially restrained columns. One such procedure²⁶ uses a modification of the Newmark method. Being based on integration it gives results of good accuracy with relatively few subdivisions of the column length. The equilibrium between the internal and the external moment is satisfied at all nodes including the end sections. The method employs both a load increment and a strain increment algorithm, the latter being used when the load approaches the failure load. The use of the strain increment algorithm allows a more accurate assessment of the failure load and also the generation of the falling branch of the load–deflexion curve. This method was improved with the introduction of a closed form solution for the concrete contribution to

Table 3

	Section	Length, in.	P_x/P_s	P_y/P_s	P_{ax}/P_s	P_{xy}/P_s	P_t/P_s	P_{xy}/P_t
Symmetric	A	150	0.827	0.644	0.981	0.574	0.610	0.941
		240	0.738	0.430	0.916	0.386	0.412	0.937
	B	150	0.841	0.589	0.974	0.541	0.586	0.922
		240	0.744	0.407	0.883	0.375	0.408	0.920
Antisymmetric	A	150	0.652	0.714	0.991	0.520	0.479	1.086
		240	0.683	0.615	0.977	0.484	0.560	0.864
	B	150	0.644	0.649	0.994	0.479	0.506	0.950
		240	0.667	0.567	0.972	0.448	0.561	0.800

All columns

6 in. $\times$ 12 in. with four bars, each 0.1963 sq. in. at 1 in. cover for section A and 1.225 sq. in. at 1.5 in. cover for section B.

For all columns

End restraints: $5(EI)_x/L$ and $5(EI)_y/L$ at both ends.

End eccentricities: 6 in. about strong axis and 3 in. about weak axis at both ends.

P_x failure load of the column under uniaxial bending about the strong axis

P_y failure load of the column under uniaxial bending about the weak axis

P_{ax} failure load under axial loading when the column is constrained to bend about the strong axis

P_{xy} failure load of the column under biaxial bending as given by the interaction formula

$$\frac{1}{P_{xy}} = \frac{1}{P_x} + \frac{1}{P_y} - \frac{1}{P_{ax}}$$

P_t theoretical failure load of the column under biaxial bending

P_s squash load of the column (107 tons for section A and 162 tons for section B).

the moment and thrust values. The computer program has been used extensively to obtain failure loads of a large number of restrained reinforced concrete columns in biaxial loading.²⁷ Apart from giving an insight into the behaviour of these columns throughout the loading range, these results were used to check the validity of an interaction formula proposed previously²⁸ for estimating the strength of biaxially loaded columns. A selection of the results is shown in Table 3 and is quite encouraging. It seems that in the absence of a better method the strength of restrained columns under biaxial bending may be estimated with good accuracy using the failure loads for restrained uniaxially loaded columns for which design charts or formulae may be more easily developed.

Dr H. R. Milner, Caulfield Institute of Technology, Melbourne

The method used by the Authors is compared with the one I used in my own work at Imperial College of Science and Technology from 1963 to 1966.^{3,17,18}

62. In Table 2 the Authors quote a theoretical failure load of 5.03 kips which I obtained for column A1. This result³ was obtained in a situation in which the component of column moment, relaxed by the effect of column shortening, was neglected. This computation was carried out to demonstrate the considerable effect that column shortening has, but it is inappropriate to compare the result with the experiments or the Author's theoretical results. Other computations reported in reference 3 show much better correlation with experiment.

63. In my work on computations on the same column in which, despite the high ratio of axial load to squash load at collapse, the member contains fibres at mid-column height which yield in tension, I decided to use either the usual $dP/d\Delta = 0$ type condition to define collapse or, for stocky heavily restrained columns, the onset of tension yield. Use of this second artificial criterion overcomes problems associated with numerically approximating high curvature gradients near and at plastic hinges. When plastic hinge conditions are being approached with increasing axial load similar problems occur. Baker *et al.*²⁹ report that plastic hinges can indeed form in certain elastically restrained columns before the occurrence of the $dP/d\Delta = 0$ condition.

64. The correct treatment of this problem is to terminate computation based on a continuous mathematical model when some selected but large curvature is reached, insert the plastic hinge, and then proceed with the discontinuous model. A reasonable value could be determined by noting when C_{y0} in the approximation $h^2 y_0^{11} = (y_2 - 2y_0 + y_{-1}) + C_{y0}$ (i.e. the high order difference corrections) becomes excessive. Of course decreasing h and iteration to include the higher order terms delays the onset of these problems but does not, in the ultimate, avoid it. For arbitrary unequal end moments the hinge or approaching hinge condition is more difficult to treat as one has to locate it first.

65. As I used $h = L/30$ for column A1 near mid-column and still experienced numerical problems of the type discussed, using the simple curvature formula $h^2 y_0^{11} = (y_1 - 2y_0 + y_{-1})$. I believe the Authors' statement in §41 is not justified. Intuitively one expects the simple curvature formula used by the Authors with $h = L/20$ or thereabouts to stiffen the column numerically. I think the Authors should examine their numerical model a little more closely.

Professor W. F. Chen, Department of Civil Engineering, Lehigh University, Pennsylvania

In a recent treatise on the theory of beam columns³⁰ the Authors' method is classified as one of the so-called 'influence coefficient methods'. I wish to put their approach in perspective and show how their numerical results may be used to extend the applicability of the design method proposed in reference 31 for steel H columns under biaxial bending to the case of steel-concrete composite beam columns.

67. The interrelationships among the variables used in beam column analysis are

shown in Fig. 15. The items enclosed by circles are known. The variables in the boxes are unknowns but they are interrelated to each other between groups. In obtaining the solution indirectly, one of the groups of variables has to be assumed first and then the other groups of variables are determined numerically. In this process new values for the initially assumed group of variables are recalculated, from which successively improved solutions can be obtained. The influence coefficient method is concerned with the development of efficient numerical procedures for this purpose.³⁰

68. There are two different approaches in the influence coefficient method (Fig. 15). One is to start from a group of variables, say $\{v_A\}$; after one cycle of calculation, it results in a new set of values for the same group of variables $\{v'_A\}$. They are then compared and used to improve the values assumed initially. The other approach starts with a group of variables $\{v_A\}$, and, after a half cycle of calculations, the values for the new variables $\{v_B\}$ are obtained. Through the other route a different set of values for the new variables $\{v'_B\}$ is calculated. They are then compared and used to improve the values assumed previously.

69. The first approach is applied by the Authors to solve a plastic steel-concrete composite beam column. In the numerical procedure the displacements u_0 , v_0 and θ_0 are first assumed at each point along the length of the member from which the corresponding curvatures (Φ_z , Φ_n , θ'_0 and θ''_0), the strain (ϵ), the stresses (σ and τ) and the internal moments (M_z , M_n and M_t) are calculated following the upper route of Fig. 15. Using these internal moments, new values for the displacements $\bar{u}$, $\bar{v}$ and $\bar{\theta}$ are calculated from the equilibrium conditions and the applied loads P , M_x , M_y and M_z . However, the newly obtained displacements $\bar{u}$, $\bar{v}$ and $\bar{\theta}$ are not used directly as the initial values for the second cycle of calculations. Rather, the modified values of displacements u , v and θ calculated through the influence coefficient matrix [ICM] equation as

$$\begin{Bmatrix} u-u_0 \\ v-v_0 \\ \theta-\theta_0 \end{Bmatrix} = [\text{ICM}] \begin{Bmatrix} \bar{u}-u_0 \\ \bar{v}-v_0 \\ \bar{\theta}-\theta_0 \end{Bmatrix} \quad \dots \quad (60)$$

are used to ascertain the convergence of solution. The detail development of [ICM] is of major concern in the Authors' work.

70. The second approach was applied by Harstead *et al.*¹⁸ on steel H columns. In this case values of curvatures (Φ_z , Φ_n , θ' and θ'') at each point along the length of the member are assumed first. The internal moments $M_{z(\text{int})}$, $M_{n(\text{int})}$ and $M_{t(\text{int})}$ are calculated through strain (ϵ) and stresses (σ and τ) as shown by the upper route of Fig. 15. Using proper boundary conditions the assumed curvatures are also integrated numerically to obtain the displacements u , v and θ from which external moments $M_{z(\text{ext})}$, $M_{n(\text{ext})}$ and $M_{t(\text{ext})}$ are calculated from equilibrium conditions and the applied loads P , M_x , M_y and M_z as shown by the lower route of Fig. 15. The internal moments and the external moments at each station are compared and their differences $\{\Delta M\}$ are used to calculate the corrections $\{\Delta \Phi\}$ for the previously assumed curvatures through the influence coefficient matrix equation as

$$\begin{Bmatrix} \Delta M_z \\ \Delta M_n \\ \Delta M_t \end{Bmatrix} = [\text{ICM}] \begin{Bmatrix} \Delta \Phi_z \\ \Delta \Phi_n \\ \Delta \theta' \\ \Delta \theta'' \end{Bmatrix} \quad \dots \quad (61)$$

71. These are general procedures for solving beam-column problems by the influence coefficient method. The actual combination of variables used in sample calculations may be different in each case for convenience in each numerical work but the Authors' method obviously belongs to this approach.

72. The interaction equations given by formulae 1.6-1(a) and 1.6-1(b) of the 1969 specification of the American Institute of Steel Construction³² are based on the limit state of instability and yielding, respectively. Research (see, for example, reference 33) has shown that these interaction equations are very conservative when compact wide

Fig. 15. Cyclic relationship of variables

flange columns are subjected to bending about both principal axes. The following ultimate strength interaction equations have been proposed³¹ and are suggested as an alternative to formulae 1.6-1(a) and 1.6-1(b) for steel columns subjected to biaxial bending

$$\left(\frac{C_{mx}M_x}{M_{ucx}}\right)^n + \left(\frac{C_{my}M_y}{M_{ucy}}\right)^n \leq 1.0 \quad (62)$$

$$\left(\frac{M_x}{M_{pcx}}\right)^c + \left(\frac{M_y}{M_{pcy}}\right)^c \leq 1.0 \quad (63)$$

Both interaction equations must be satisfied for any biaxially loaded beam column.

73. In equations (62) and (63) C_{mx} and C_{my} are equivalent moment coefficients defined as in section 1.6.1 of reference 32; M_x and M_y are factored maximum end moments at the ends of the beam column about the x and the y axes, respectively; working load moments are to be multiplied by the load factors 1.7 or 1.3, depending on whether only gravity loading or gravity and wind or earthquake loading, respectively, is present. M_{ucx} and M_{ucy} are ultimate symmetrical single curvature moment capacities of the beam column about the x and y axes, respectively, when there is zero moment about the other axis. M_{pcx} and M_{pcy} are plastic moments about the x or y axes, respectively, reduced for the presence of axial load.

$$M_{ucx} = M_{ux} \left(1 - \frac{P}{P_u}\right) \left(1 - \frac{P}{P_{ex}}\right) \quad (64)$$

$$M_{ucy} = M_{uy} \left(1 - \frac{P}{P_u}\right) \left(1 - \frac{P}{P_{ey}}\right) \quad (65)$$

$$M_{pcx} = 1.18 M_{px} \left(1 - \frac{P}{P_y}\right) \leq 1.0 \quad (66)$$

$$M_{poy} = 1.19 M_{py} \left[1 - \left(\frac{P}{P_y} \right)^2 \right] \leq 1.0 \quad (67)$$

$$\eta = 0.4 + \frac{P}{P_y} + \frac{b_t}{d} \geq 1.0 \text{ when } b_t/d \geq 0.3 \quad (68)$$

$$\eta = 1.0 \text{ when } b_t/d < 0.3 \quad (69)$$

$$\xi = 1.6 - \frac{P/P_y}{2 \ln(P/P_y)} \quad (70)$$

74. In equations (64)–(70) P is factored axial load, i.e. the working axial force multiplied by a load factor of 1.7 or 1.3, as appropriate; P_u is the ultimate capacity of the beam column member in the absence of bending moment; $P_u = 1.7 A F_a$, where A is the area and F_a is the allowable compressive stress as defined in section 1.5.1.3 of reference 32 for the largest effective slenderness ratio. P_{ex} and P_{ey} are the elastic buckling load of the beam column member in the absence of bending moment about the x and y axes, respectively; $P_e = (23/12) A F'_e$ where F'_e is as defined in section 1.6.1 of reference 32. P_y is the yield load in the absence of moment $P_y = A F_y$. M_{px} and M_{py} are the plastic moment capacities about the x and y axes, respectively; $M_p = Z F_y$ where Z is the plastic section modulus. M_{ux} and M_{uy} are the moment capacities about the x and y axes, respectively, when only one moment acts and the axial force is zero because only compact shapes are allowed for in this procedure (i.e. the flange and web width thickness provisions of section 1.5.1.4.1 of reference 32 b and d must be satisfied); $M_{ux} = M_{px}$ and $M_{uy} = M_{py}$ except that M_{ux} must be reduced to account for the presence of lateral-torsional buckling, when necessary, according to the formula

$$M_{ux} = M_{px} \left[1.07 - \frac{(L/r_y) \sqrt{F_y}}{3160} \right] \leq M_{px} \quad (71)$$

where L is the unbraced length.

75. These interaction equations are ultimate strength equations and it may be possible that some yielding is present under working loads. In case such yielding must be avoided; formulae 1.6–1(a) and 1.6–1(b) of reference 32 should also be checked under working loads. These ultimate strength interaction equations should not be used for frames subject to joint translation (sidesway) where reference 32 recommends $C_m = 0.85$ and the effective lengths are larger than unity. However, they may be used for such frames when the forces are determined by second order analysis, i.e. the $P-\Delta$ effects at ultimate load are considered in the structural analysis. These ultimate strength interaction equations are not intended for plastically designed frames in which plastic hinges form in the column and where forces redistribute themselves as the ultimate load is reached.

76. This design method has been extended to the case of reinforced concrete beam columns under biaxial loading.³⁴ Against the background of this information, it is of great interest to see that the Authors will apply their computer model to study the suitability of the design method mentioned here in the case of steel-concrete composite beam columns.

Dr Virdi and Dr Dowling

Our work was concerned mainly with composite steel-concrete columns. As pointed out in the Paper, the method presented is therefore applicable mainly to torsionally stiff sections. The reason for selecting test results on bare steel H sections for comparison with our analytical results was that test results for biaxially restrained composite or reinforced concrete columns in biaxial bending were not available.

78. We agree with Dr Vinnakota and Dr Sen that the number of results presented for steel H sections is insufficient for general conclusions on the behaviour of bare steel columns to be drawn. We also agree that it would be necessary to account for torsional effects in certain torsionally weak columns. However, the good agreement between

Fig. 16. Deflexions

experimental and theoretical results obtained by the present method clearly indicates that it is possible to predict the strength of steel columns with cross-sections which are often regarded as torsionally weak by applying inelastic analyses in which torsional effects are neglected. The practical implications are that in many cases of bare steel column construction torsional buckling may not be as serious a consideration as research papers sometimes suggest.

79. The heavy axial load condition may not, as mentioned by Dr Vinnakota, include all loading types likely to occur in real columns, but it is an important loading condition for biaxially restrained H columns in multistorey structures. This is particularly true when, as is commonly the case, they are braced against sideways. For short to medium length columns, such as those tested by Gent and Milner,¹⁷ the errors in ultimate loads obtained by ignoring torsion would be within acceptable limits (under 5%).

80. For long restrained columns with rotational end restraints the errors in ultimate loads will also be small. The column analysed by Dr Sen may be considered as belonging to this classification. It is instructive to compare the results obtained by Dr Sen with the results in Fig. 16. Only the case with weaker beam restraints is considered. The unbroken lines refer to the case without torsion and the broken lines to the case with torsion. The error in ultimate moments is less than 1%. The major axis deflexions in the two cases are indistinguishable. Minor axis deflexions are significantly different near collapse, but this does not appear to have influenced the ultimate strength. However, for the more rare unrestrained long column the errors obtained by ignoring torsion could be more significant.

81. Dr Basu's contention that integration techniques are likely to be more accurate than the differentiation techniques seems unsubstantiated as, to our knowledge, no comprehensive comparisons have been made so far.

82. Some of the other points he raised do not appear to be based on facts. As outlined in the Paper, by adopting a multiplying factor of 0.1, the number of times the trial load exceeds the ultimate load can be effectively reduced. Thus, to obtain an ultimate load accurate to within 0.1% of the squash load, the number of times the trial load will exceed the ultimate load is only three, provided the initial increment is 0.1 times the squash load.

83. Dr Basu is mistaken in stating that our method does not allow the monitoring of the descending branch of the load-deflexion response. The post peak characteristics have been obtained successfully for over 160 biaxially restrained reinforced concrete

columns.^{35,36} The only requirement is that the decrement in the value of the trial load in the descending zone has to be kept very small. The incremental strain approach suggested by Dr Basu may be useful in reducing the number of points required to monitor the curve adequately.

84. The reservation Dr Basu mentions regarding the equilibrium conditions at the two ends is also unfounded. Closer scrutiny of the equations should allay his worries.

85. If the column is divided into $n+1$ segments, the number of unknowns in the problem will be $2n$ excluding the fictitious points. The equations for these $2n$ unknowns obtained from a consideration of equilibrium at the n internal points are identical in the two cases. Except in the case of restrained columns the equations for the first and the last internal points will not include terms relating the fictitious points. The four fictitious unknowns are related to the four conditions of equilibrium obtained at the ends (two at each end). A partitioning of matrices between the internal unknowns and the external unknowns will result in exactly the same equations for the internal unknowns as used in our method. The additional conditions of equilibrium merely give the values for the fictitious points. It will thus be clear that, at least for the case of pin-ended columns, there will be no difference in the two solutions. For the restrained columns, the only source of error arises from the expression used for end slopes. It may be seen that the expressions used in the Paper are equivalent to backward and forward difference formulae (depending on which end of the column is in question). By taking more backward or forward points the estimate of the slope can be improved although we do not think this is necessary. This is adequately confirmed by the results shown in Fig. 16, where the slopes obtained by Dr Sen were calculated on the basis of fictitious points and can be seen to differ little from those we obtained. The differences in the weak axis slopes near collapse are due more to the interaction with the torsional failure mode than to any other factor. In spite of this difference in slopes, the failure load is practically the same in both cases. Thus it would appear that Dr Basu's contention regarding the use of fictitious points is untenable. In the method based on integration he used there is no alternative but to include the fictitious points, but this is not a restriction of our method.

86. We are grateful to Dr Basu and Professor Chen for the design equations mentioned. We examined the biaxial interaction formula mentioned by Dr Basu for reinforced concrete sections in connexion with composite columns in an earlier paper¹ and found it to be conservative in all cases. Although many biaxial interaction formulae relate biaxial strength to the strength of columns constrained to fail with uniaxial bending about the major or minor axis, this interaction formula allows for the influence of minor axis imperfections when only major axis moments are present. A study we have nearly completed shows how this effect can be more accurately accounted for by modifying this interaction formula. The interaction formulae mentioned by Professor Chen need further examination when applied to composite and/or reinforced concrete columns. They will, almost certainly, prove to be conservative because of the beneficial effect of concrete encasement on the stability of I or H shapes.

87. The points raised by Dr Milner are not very clear. Contrary to his claim that the use of first tension yield as a failure criterion overcomes the problem created by the development of a plastic hinge, the results presented in the Paper show that even without making special provisions for high curvatures at the plastic hinges, our procedure gives results that agree better with tests. By reducing the spacing between nodes it is possible to improve the estimate of curvatures near the points of plastic hinge. In the general case of biaxial bending with unequal end moments at the two ends, however, the location of maximum strong axis curvature will not necessarily coincide with either the location of maximum weak axis curvature or with the location of maximum biaxial curvature. In such an event an equal spacing would appear to be the best alternative. Our method can easily be modified to allow for unequal spacings. In this connexion we would also like to refer Dr Milner to § 49 where Dr Vinnakota states that his findings agreed with ours as regards the requisite number of column subdivisions. It is possible that the numerical difficulties Dr Milner experienced were not related to the choice of

the number of segments into which the column was divided, but may relate to the fact that his method, being a first order method, is by its very nature slow to converge. In our method the convergence is speeded up considerably as due account is taken of the rate of change of calculated deflexions, not only at the points in question but also at their neighbouring points. There are several procedures available for increasing the rate of convergence of first order methods (e.g. Aitken's Δ square method³⁷) but as they apply only to individual points and do not take into account the values at the neighbouring points, the convergence is not as fast as the Newton-Raphson method.

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