A note on the discharge of critical depth measuring devices with arbitrary shape

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The Author has referred to several reports by myself and Mr Harrison on flumes and other critical depth gauging structures, but there is little difference between the Author's results and the methods currently in use at Wallingford. Another useful reference is the discussion^{1,2} of the paper by Hall.³ It contains several valuable contributions to the development of flow measurement. I included a derivation of the discharge equations, similar to that given in Appendix IV of reference 8, which followed a suggestion from Professor C. M. White. Basing the treatment on equations referring to the zone of potential flow outside the boundary simplifies the algebra, without in any way detracting from the validity of the result. Applying the critical depth principle of maximum discharge to these equations avoids the complexities arising from non-uniform velocity distribution if equations relating to the total depth, including the boundary layer, are used. Approaching the problem this way avoids the necessity for incorporating a velocity distribution coefficient in the analysis. The method (Appendix IV, reference 8) is as follows.

49. Outside the boundary layer, the velocity is given by:

$$U = \sqrt{2g(H_p - D)} \quad \dots \dots \dots (18)$$

(The Author's nomenclature is used, except that for clarity H_p is used to denote the total head at any point in the zone of potential flow.) From the definition of the boundary layer displacement thickness, the continuity equation can be written:

$$Q = (A - P\delta_d)U \quad \dots \dots \dots (19)$$

Hence:

$$Q = \sqrt{2g}(A - P\delta_d)(H_p - D)^{1/2} \quad \dots \dots \dots (20)$$

50. Differentiating with respect to D , setting $dQ/dD=0$, and noting that $dA/dD=B$

$$(H_p - D) = \frac{A - P\delta_d}{2\left(B - \frac{dP}{dD}\delta_d\right)} \quad \dots \dots \dots (21)$$

Inserting into equation (18), and then solving equation (19):

$$Q = (A - P\delta_d) \left\{ \frac{g(A - P\delta_d)}{\left(B - \frac{dP}{dD}\delta_d\right)} \right\}^{1/2} \quad \dots \dots \dots (22)$$

Also:

$$H_p = D + \frac{Q^2}{2g(A - P\delta_d)^2} \quad \dots \dots \dots (23)$$

51. This pair of equations provides a parametric basis of calibration for any critical depth service. The total head in the potential zone outside the boundary layer H_p , is identical with the total head that exists just upstream of the contraction, which the Author denotes by H_0 , and calls original specific head.

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52. Equations (22) and (23) may be written:

$$Q = A_* \sqrt{\left(\frac{g A_*}{B_*} \right)} \quad \dots \dots \dots (24)$$

$$H_0 = D + \frac{1}{2} \frac{A_*}{B_*} \quad \dots \dots \dots (25)$$

where the suffix $*$ denotes the area and width at the critical section assuming that the boundaries are displaced inwards by δ_a . Thus the head to discharge relationship for a given throat geometry can readily be deduced.

53. It was shown⁶ that these equations implying displaced boundaries could, with some algebraic manipulation, be replaced by an alternative pair of equations identical with the Author's equations (13) and (16).

54. These are applied by working out the head to discharge relationship assuming an ideal fluid and the actual throat dimensions, and then adjusting the upstream total head by the amount $(P/B)\delta_a$. Both these approaches will be found in the forthcoming British Standard, and of course they give the same answers apart from second order effects arising from the product of small first order terms. Which is the more convenient in a particular case depends on the computational facilities available and on the shape of flume under consideration.

55. Thus the Author's work provides very timely confirmation of the principles used in the draft British Standard for rectangular, trapezoidal and U shape flumes, and in the corresponding UK submission to the International Standards Organization. If doubts remain, they are not in the basic theory and equations, but in the reliability of the estimation of δ_a , the boundary layer displacement. At present, Fig. 3 of reference 5 appears to be the best information available, but a study of that figure shows that δ_a is dependent on the Reynolds number at which transition from a laminar to a turbulent boundary layer occurs. Because there was an unfortunate arithmetic error in the computation, a new version of the diagram has been prepared (Fig. 7). It can be seen that at low Reynolds numbers (i.e. small laboratory scale flumes) the boundary layer will be laminar throughout, and there is not much doubt in the estimate of δ_* , except that which arises from the application of results for non-accelerating flow to the accelerating flow in the throat of a flume. At the other extreme, when the Reynolds number is above 10^7 (very large field installations) the estimate again becomes insensitive to the point of transition. In the zone $5 \times 10^5 < R_L < 5 \times 10^6$, the estimation of δ_a depends appreciably on the assumed value of $R_{L,trans}$. It seems appropriate in engineering installations, for example concrete structures, to assume $R_{L,trans} = 3 \times 10^5$, but in smooth installations, especially those with a gently streamlined entrance transition and good approach conditions, the transition will be delayed. There is evidence of this from the V shape critical depth device studied by Smith and Liang.¹³

56. In Table 1, the experimentally determined coefficient values for a V shape throat are compared with those deduced theoretically, first with $R_{L,trans} = 3 \times 10^5$ and then with $R_{L,trans} = 10^6$. There is no significant difference with the shorter throat lengths, because R_L is too small to have much effect on δ_a . But with a throat length of 11 ft, the assessment of δ_a has an important effect on the coefficient, and the value of R_L is in the difficult region. The assumption of $R_{L,trans} = 3 \times 10^5$ was found to give coefficients appreciably below those measured, while coefficients based on $R_{L,trans} = 10^6$ showed close agreement, the discrepancy averaging less than 1%.

57. Although recent researches, both theoretical and experimental, have provided improved understanding of flow measurement using critical depth devices, there remains considerable scope for further study. In addition to obtaining data from accurate calibrations of flumes over a range of sizes, it is apparent that research is needed on boundary layer development in the accelerating flow of the throat of a flume.

58. However, it must be borne in mind field devices operate at higher values of R_L than can be covered in laboratory installations. Moreover, they are not usually

Fig. 7

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hydraulically smooth. Laboratory research on boundary layer development may therefore be of limited application to prototype structures.

Mr A. J. M. Harrison, Hydraulics Research Station, Wallingford

In § 32 of the Paper the Author derives an equation for critical depth in a rectangular-section measuring device. (See corrigenda on page 510):

$$D_o = \frac{2}{3} H_o \left(1 + \frac{1}{3} \epsilon - \frac{4\delta_a}{3b} - \frac{\delta_a}{H_o} \right) \quad \dots \quad (26)$$

He compares this equation with the equation derived in my earlier Paper:⁵

$$D_o = \frac{2}{3} H_o \left(1 + \frac{\delta_a}{2H_o} \right) \quad \dots \quad (27)$$

However, equation (27) holds for any aspect ratio and not just for the case $b/D \rightarrow$ infinity as the Author suggests.

60. The Author is correct in noting that the two equations do not agree. He suggests that this is because of a less justified use of the minimum specific energy principle in deriving equation (27). In fact in my paper I used the maximum discharge principle, setting $dQ/dD=0$ for a given upstream head H_o . In the discussion of my paper,¹⁴ Williams pointed out that it would be preferable from an idealistic viewpoint to apply the maximum discharge principle for a given head at the crest. However, he showed that in an analysis to first order in δ_a and δ_o the same discharge equation resulted. The energy thickness δ_o is defined as:

$$\delta_o = \frac{1}{P} \int \frac{u}{U} \left\{ 1 - \left(\frac{u}{U} \right)^2 \right\} dA \quad \dots \quad (28)$$

61. Since the maximum discharge and minimum energy principles should lead to the same result, it is interesting to know why equations (26) and (27) differ. The fact that I held the upstream head constant in the differentiation is only part of the explanation; in addition it appears that in the derivation of equation (11) by the minimum energy principle the Author has made the incorrect assumption that α^1 does not vary with D . Because α^1 is used to determine the average velocity head of

Table 1. Comparison of theory and experiment: triangular throat.¹³ (Throat assumed to be hydraulically smooth)

Throat length, L , ft	$\frac{H}{L}$	Throat Reynolds number R_L , 10^5	Calculated discharge, Q , cusecs		Calculated coefficient C_D		Experimental value of C_D
			$R_{L, trans}$ 3×10^5	$R_{L, trans}$ 10^6	$R_{L, trans}$ 3×10^5	$R_{L, trans}$ 10^6	
1.33	0.60*	3.52	1.280	1.279	0.983	0.982	0.995*
1.33	0.30	2.49	0.219	0.219	0.958	0.958	0.965
1.83	0.40	4.64	1.029	1.024	0.980	0.976	0.982
1.83	0.20	3.28	0.174	0.174	0.947	0.946	0.969
2.33	0.30	5.78	0.910	0.907	0.975	0.972	0.976
2.33	0.15	4.09	0.155	0.153	0.945	0.936	0.965
11.00	0.024	17.1	0.0578	0.0705	0.689	0.812	0.795
11.00	0.029	18.7	0.0991	0.1145	0.739	0.832	0.828
11.00	0.039	21.6	0.225	0.246	0.803	0.861	0.867
11.00	0.049	24.2	0.417	0.442	0.842	0.883	0.893
11.00	0.059	26.5	0.684	0.714	0.869	0.900	0.910

* High value of H/L may affect C_D .

the whole flow section the proportion of the section which is occupied by the boundary layer affects the value of α^1 . On the other hand, the boundary layer parameters δ_a and δ_o can be considered constant at a section since they are independent of the velocity U outside the boundary layer (and hence the depth) in rough turbulent flow and vary only very slowly with U in laminar, smooth turbulent or turbulent transitional flow.⁶

62. The minimum specific energy approach should therefore be applied. The continuity equation is:

$$Q = \{bD - (b + 2D)\delta_a\}U = \bar{u}bD \quad . \quad . \quad . \quad (29)$$

Using equations (28) and (29) and the equations of § 24 the specific head equation becomes:

$$H = D + \frac{Q^2}{2gb^2D^2} \frac{\{1 - (\delta_a/D) - (2\delta_a/b) - (\delta_o/D) - (2\delta_o/b)\}}{\{1 - (\delta_a/D) - (2\delta_a/b)\}^3} \quad . \quad . \quad . \quad (30)$$

Differentiating equation (30) and setting $dH/dD = 0$:

$$D_o^3 = \frac{Q^2}{gb^2} \left[\frac{1 - (\delta_a/D_o)\{1 + (2D_o/b)\} - (3\delta_o/2D_o)\{1 + (4D_o/3b)\}}{\{1 - (\delta_a/D_o)\}^4\{1 - (2\delta_a/b)\}^3} \right] \quad . \quad (31)$$

Applying equations (1) and (2) to rectangular sections:

$$Q^2 = 2g(H_o - D_o)\{bD_o - (b + 2D_o)\delta_a\}^2 \quad . \quad . \quad . \quad (32)$$

Hence from equations (31) and (32):

$$D_o = \frac{2}{3}H_o \left\{ 1 + \frac{\delta_a}{2H_o} - \frac{3}{4}\frac{\delta_o}{H_o} \left(1 + \frac{8}{9}\frac{H_o}{b} \right) \right\} \quad . \quad . \quad . \quad (33)$$

to first order in δ_a and δ_o . The result is thus similar to equation (27) except for the inclusion of the δ_o/H_o term. The latter term is approximately equal to δ_a/H_o for turbulent boundary layers. Thus the critical depth D_o is slightly less than $2/3$ if one prefers Williams's application of the principle of maximum discharge.

63. In practice, as the Author shows, small differences in the deduced value of D_o do not affect the discharge equation (17). A more important point is the actual value of δ_a used in the discharge computation. This is generally calculated from theory based on the assumption of a flat plate in an infinite fluid without a pressure gradient. In fact the pressure gradient at the bed is initially adverse and then favourable (see Fig. 7 of reference 5); upstream conditions and the depth of flow may also influence the growth of the boundary layer; and the location of the position at which critical conditions occur is still in doubt. There seems no reason, therefore, to change the conclusion I reached in replying to the discussion on broad-crested weirs:¹⁴ only further large-scale experiments in which the actual growth of the boundary layer is measured can decide whether the simple method at present in use needs modification.

Mr J. A. Perkins, Hydraulics Research Station, Wallingford

I have read the Paper with interest, although I must admit to having experienced some confusion with the notation. For example, H^* , which is the specific head without any allowance for boundary layer effects, i.e. the specific head for an ideal fluid, is termed the corrected original specific head. This implies that H_o , the original specific head, is in some way incorrect, whereas in fact it is H^* that is in need of correction. It would possibly have been clearer if H^* had been called the specific head for an ideal fluid and H_o the specific head, corrected for boundary layer effects.

65. Some uncertainty also arose from the use of D_1 as the approximate critical depth in the throat. The Paper did not bring out sufficiently clearly the relationship between the various methods by which D_1 was calculated and the boundary layer approach discussed in the early part of the Paper. The correction to the upstream specific head in the Author's equation (16), necessary as a result of boundary layer

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effects, is identical with equation IV.21 in Appendix IV of reference 8. This correction represents the additional head necessary to discharge a real fluid compared with the head required to discharge an equal quantity of ideal fluid: the discharge rate is equal to $(gA_c^3/B_c)^{1/2}$ where A_c and B_c are the area and water surface width at the critical section in the throat, for an ideal fluid. However, as a result of boundary layer growth, the actual depth in the throat for this discharge will be greater than D_c . Although the Author states in § 29 that the accurate value of critical depth is not important, the effect of the boundary layer on the depth in the throat can have some significance. This can be shown by taking the triangular flume as an example. If equation (16) is used to determine H_0 , the upstream specific head, assuming that D_1 is the depth in the throat, then the coefficient of discharge of the triangular flume will be:

$$C_{DA} = \frac{1}{\left(1 + \frac{4\delta_a}{5D_1} \sin \phi\right)^{5/2}}$$

66. Alternatively if the upstream energy is calculated from equation (3), allowing for the boundary layer displacement thickness δ_a in the throat (still assuming that the overall depth is D_1), then the coefficient of discharge is:

$$C_{DB} = \left[\frac{1 - \frac{\delta_a}{D_1 \sin \phi}}{1 - \frac{\delta_a}{5D_1 \sin \phi}} \right]^{5/2}$$

67. Table 2 shows the ratio of the two discharge coefficients for different values of $(\delta_a/D_1 \sin \phi)$. Thus for $(\delta_a/D_1 \sin \phi) = 0.10$, which is not an out of the way value, the neglect of the boundary layer thickness in the throat can lead to an overestimate of the discharge coefficient by 2%.

68. As mentioned previously, the depth in the throat of a flume will be greater than D_c , the critical depth for an ideal fluid. The extent of the difference can be found from a consideration of the specific energy. For a rectangular flume the upstream specific energy (from equation (16)) is:

$$H_0 = \frac{3D_c}{2} + \frac{P_c \delta_a}{B_c}$$

From equation (3), taking account of the presence of the boundary layer:

$$H_0 = D + \frac{Q^2}{2g(A - P\delta)^2}$$

where D is the actual depth in throat.

69. This has been shown by Mr Ackers to be equivalent to:

$$H_0 = D + \frac{1}{2} \frac{A_+}{B_+}$$

Table 2

$\frac{\delta_a}{D_1 \sin \phi}$	$\frac{C_{DB}}{C_{DA}}$
0.01	0.9999
0.05	0.9950
0.10	0.9797
0.15	0.9543

for the general case, where A_+ and B_+ refer to the net section, after allowing for the boundary layer displacement thickness δ_a . For a rectangular flume this simplifies to:

$$H_0 = D + \frac{1}{2}(D - \delta_a)$$

Equating the two expressions for H_0 gives:

$$\begin{aligned} D &= D_c + \delta_a \left(\frac{1}{3} + \frac{2}{3} \frac{P_c}{B_c} \right) \\ &= D_c + \delta_a + \frac{4D_c\delta_a}{3B_c} \\ V &= \sqrt{g \left(D_c + \frac{4}{3} \frac{D_c}{B_c} \delta_a \right)} \end{aligned}$$

70. The relationship between the upstream specific energy and the depth in the throat can be seen by considering the alternative expressions for H_0 . If the coefficient α^1 is introduced into equation (16), in order to be consistent with §§ 24 and 25, then

$$H_0 = D_c + \frac{D_c}{2\alpha^1} + \frac{P_c}{B_c} \delta_a$$

71. Substituting $(\epsilon + 1) = \alpha'$, leads to:

$$D_c = \frac{2}{3} H_0 \left(1 + \frac{\epsilon}{3} - \frac{4\delta_a}{3B_c} - \frac{\delta_a}{H_0} \right)$$

which is identical with the equation in § 32 (see corrigenda). However if H_0 is considered in terms of D , the actual depth in the throat, and not in terms of D_c , the critical depth for an ideal fluid, then:

$$H_0 = D + \frac{1}{2}(D - \delta_a)$$

and:

$$D = \frac{2}{3} H_0 \left(1 - \frac{\delta_a}{2H_0} \right)$$

which is Harrison's equation for the critical depth.

Professor B. C. Yen and Mr A. S. Sevuk, University of Illinois

The Author is to be commended for his effort to introduce once again the boundary layer displacement and momentum thicknesses in order to improve the accuracy of one-dimensional non-uniform open-channel flow equations when applied to critical depth measuring devices. The one-dimensional flow equations are obviously poor approximations to curvilinear flows with high convective acceleration such as the present case in the neighbourhood of critical flow region. Therefore, any technique which would improve the accuracy yet still preserve the simplicity of the one-dimensional approach is surely welcome. Nevertheless, it should be noted that for any cross section the same information on velocity distribution is needed to evaluate δ_a and δ_m and the Boussinesq coefficient α and Coriolis coefficient α^1 .

73. In § 14 the Author brought out an interesting point of the difference of the discharge relationships derived based on three different criteria. In a paper by Yen and Wenzel¹⁵ it was pointed out that the momentum approach is inherently different from the energy approach and the corresponding dynamic equations are different but related. In particular, the difference between momentum and energy approaches is expressed in terms of the variation of α and α^1 and the deviation of the pressure distribution from hydrostatic. In the highly curvilinear flow region around the critical section, a hydrostatic pressure distribution is hardly justified. For the horizontal channel case studied by the Author, the difference among the discharge relationships can apparently be accounted for by the pressure distribution and the 'minor' terms being neglected.

J. P. Th. Kalkwijk

Having read the discussions of Messrs Ackers, Harrison and Perkins, Professor Yen and Mr Sevuk with much interest I have the feeling that one of the essential meanings of my remarks has not come over. That is why I would like to emphasize one of my conclusions, in a modified form: discharge formulae for the measuring devices considered do not depend on the method applied, but the critical depth does. The criteria used here were derived from the long wave equations and the minimum specific energy principle. As long as the correction of the critical depth for the ideal fluid case remains small, the discharge relationship does not change. If other assumptions are used, such as those of Mr Ackers (δ_a not depending on D) and Mr Harrison (δ_a and δ_e not depending on D) the intermediate steps may be different, but the final result is the same.

75. Nevertheless, there remains the difficult problem of the determination of the displacement thickness. Some provisional computations for a weir with infinite width, based upon recent developments of boundary layer theory, show a considerable decrease of the boundary layer thickness with respect to the case of the plate in an infinite fluid.

76. All methods, however, assume nearly horizontal flow on the weir, but this is not justified at the end. At the point where critical depth occurs the streamlines are highly curved and vertical accelerations are not negligible. Nevertheless, it is believed that the relationships derived hold, provided the weir is long enough. However, it should be noted that the critical depth, independent of the method applied, is a virtual quantity which cannot be denoted in the physical model. In fact it may be concluded that the nearly horizontal flow assumption is rather poor.¹⁵

77. With regard to the remarks of Mr Perkins I would like to say that only the discharge coefficient C_{DA} can be derived from my remarks. For this purpose equations (13) and (16) can be used; the result should be obtained by means of the depth D_1 , which differs from the critical depth D_c .

78. Furthermore, I agree there is some confusion concerning the specific heads H_0 and H^* . At best this may be demonstrated by the last equation of § 71 which is not in agreement with equation (27).

Corrigenda

In § 34 for $P_1 = 2D_1 \cos \phi$ read $P = 2D_1/\cos \phi$

Hence for $H_0 - \frac{\cos^2 \phi}{\sin \phi} \delta_a = \frac{5}{4} D_1$ read $H_0 - \frac{1}{\sin \phi} \delta_a = \frac{5}{4} D_1$

and for the equation in § 35 read

$$Q = \left(\frac{4}{5}\right)^{5/2} \frac{1}{2} \sqrt{2g} \tan \phi \left(H_0 - \frac{1}{\sin \phi} \delta_a\right)^{5/2}$$

For the second mathematical equation in § 32 read

$$D_e = \frac{2}{3} H_0 \left(1 + \frac{1}{3} \epsilon - \frac{4\delta_a}{3b} - \frac{\delta_a}{H_0}\right)$$

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