Water surface profiles in gradually varied open-channel flow

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The Paper is a valuable contribution to the literature of open-channel flow and is of great interest on both the theoretical and experimental aspects of the problem. The conclusions tabulated in § 105 should be immediately useful to designers in their choice of a method of computation appropriate to different cases.

111. I wonder whether the Authors have also considered the example of an open channel of *circular* cross section and, in particular, whether they would like to comment on the case analysed by the late Professor A. H. Gibson.²⁰ Here, Gibson applied the 'backwater function' for a *wide* channel to 'a circular concrete conduit 7.30 ft in dia., having a gradient of 1 in 3168 or 20 in. per mile, discharging 30 million gallons per day at a normal central depth of 3.40 ft.' The depth was increased to 5.48 ft by means of a weir, and Gibson estimated that the new depth 4460 ft upstream would be 4.52 ft. In doing so, he used the value of 3.40 ft, i.e. the original central depth with uniform flow, in the same way as if the channel had been of wide rectangular section with such an initial constant depth. He did, however, determine the value of Chezy's *C* from the given data for the circular section: cross-sectional area 19.2 sq. ft, wetted perimeter 10.9 ft, hydraulic mean depth 1.76 ft, from which it followed that the mean velocity for uniform flow was 2.90 ft/s and correspondingly ' $C = v/\sqrt{(mi)} = 123.5$ ' or, say, 124.

112. Gibson stated that A. H. Jameson had worked out an 'accurate solution' for this particular case and had plotted the surface profile, which indicated that the depth 4460 ft from the weir would be 4.50 ft. Gibson argued, therefore, that if the value of the depth for uniform flow is known for the actual channel, the results obtained by using the 'backwater function' as tabulated for a wide rectangular channel in which the hydraulic mean depth is sensibly the same as the actual depth 'are very nearly in agreement with those obtained by more cumbersome methods'—even in the extreme case of a channel of circular cross section.

Mr A. A. Smith, University of Strathclyde

My interest in the computation of steady, gradually varied flow arises partly from a teaching viewpoint, partly as a result of a number of *ad hoc* investigations, but principally from an interest in the application of digital computer techniques to problems of civil engineering hydraulics. As pointed out by the Authors, it is always possible to obtain a solution by numerical integration methods employing a computer. In view therefore of the increasing availability of time-sharing multi-access facilities it is, in my view, becoming more and more difficult to make a convincing case for the practical application of analytical methods which have restricted applicability. On a number of occasions²¹⁻²³ I have advocated the use of a library of procedures, each representing a basic computational routine, for the easy and economical development of programs of a more sophisticated or complex nature. The analysis of gradually varied flow is a case in point where, without difficulty, one may break down the problem into a series of 'building blocks' thus providing the flexibility which is so essential if the application of computer techniques to civil engineering is not to be

Table 2. Specification of computational routines

PROPERTIES (*B, H, PTS, WL, A, T, P, AY*): Finds the area *A*, topwidth *T*, wet perimeter *P* and product of area times centroidal depth *AY* for the specified water level *WL* in a channel of arbitrary cross section defined by the straights joining the points—*PTS* in number—the co-ordinates of which are stored in the arrays *B, H[1:PTS]*.

INVERT (*H, PTS, IL, WLMAX*): Finds the invert level *IL* and the maximum permissible water level in a cross section defined by the co-ordinates stored in the arrays *B, H[1:PTS]*.

STRICKLER *Q(A, P, S, K, G)*: Calculates the normal discharge by means of the Strickler formula for the specified conditions of area *A*, wet perimeter *P*, bed slope *S*, roughness height *K* and gravitational acceleration *G*. (This function procedure is one of a series with identical formal parameters which may be called by the dummy identifier 'NORMAL *Q*').

SFROMQ (*QSPEC, A, P, K, G, NORMAL Q*): Calculates the slope of the energy line for a specified discharge *QSPEC* in a section of area *A*, perimeter *P* with gravitational acceleration *G*. The resistance to flow is defined by the normal discharge function *NORMAL Q* together with the roughness measure *K* which must be compatible with *NORMAL Q*.

GVF FUNCTION (*B, H, PTS, Q, S, K, NORMAL Q, G, Y*): Calculates the value of the gradually varied flow function dx/dy for the section defined by the co-ordinates in the arrays *B, H[1:PTS]* with discharge *Q*, bed slope *S*, gravitational acceleration *G*, when the depth is given by *Y*. The resistance to flow is defined by the normal discharge function *NORMAL Q* together with the appropriate roughness measure *K*. Uses *PROPERTIES*, *INVERT*, *SFROMQ* and the desired function to replace *NORMAL Q*.

abused. It is, for instance, very easy to create a procedure which enables the properties of an arbitrary cross section to be generated for any specified water level. More important, in my opinion, it is equally easy with the procedural languages in common use nowadays to allow the user to choose whichever resistance law he may consider to be appropriate, irrespective of whether this is amenable to mathematical expression or described in the form of experimentally determined data. These and other procedures are specified in Table 2 and can be simply combined by a very short driving program to compute gradually varied flow profiles in any prismatic channel for any set of conditions.

114. Although the numerical integration of surface profiles may be automated without difficulty, there is occasionally doubt as to the size of step which should be used. In § 49 the Authors mention the testing of step size for the Runge-Kutta method adopted by the simple expedient of halving the increment. For the engineer in need of an answer immediately this method is quite satisfactory when a time-sharing facility is employed. When batch processing systems are used, however, there is some need for guidance before committing the data for processing. The necessary step size depends on the desired accuracy and also on the choice of method for averaging the function or components thereof over the elementary reach. For instance, the graphical method mentioned by the Authors offers an obvious proof that the evaluation of the function $\phi(y) = dx/dy$ for the mean depth $(y_1 + y_2)/2$ is more accurate than the averaging of the two values of the function at the reach extremities.

An even better approximation is given by using Simpson's Rule:

$$(x_2 - x_1) = (y_2 - y_1) \frac{1}{6} [\phi(y_1) + 4\phi(y_m) + \phi(y_2)]$$

115. To illustrate this point, I have carried out numerical integration of simple profiles by various methods and compared the results obtained with different step sizes with the results obtained by analytical integration. To facilitate the latter, two elementary cross sections were selected, namely,

- (a) a rectangular, semi-infinite channel of zero slope;
- (b) a triangular channel with 1:1 side slopes and zero bed slope.

Using a resistance equation of the Manning-Strickler form, i.e.

$$V = Q/A = \frac{8.41(g)^{1/2}}{k^{1/6}} m^{2/3} i_t^{1/2}$$

the equation of the surface profile may be obtained by direct integration as follows.

Case (a):

$$x = C_1 y^{4/3} - C_2 y^{13/3}$$

where

$$C_1 = (3/4)(8.41)^2/k^{1/3}$$

and

$$C_2 = (3/13) \frac{(8.41)^2 g}{k^{1/3} Q^2}$$

Case (b):

$$x = C_3 y^{4/3} - C_4 y^{19/3}$$

where

$$C_3 = (3/8)(8.41)^2/k^{1/3}$$

and

$$C_4 = (3/76) \frac{(8.41)^2 g}{k^{1/3} Q^2}$$

116. In each case the roughness height used was 0.01 ft and the computed profile was the H2 curve from a downstream control depth of $y = y_{cr} = 1.0$ ft. With the latter condition, the discharge is defined as $q = (g)^{1/2}$ and $Q = (g/2)^{1/2}$ for cases (a) and (b), respectively.

117. Two methods for determining the numerical integral were employed; one method was based on the Simpson's Rule equation given above, while the other consisted simply of the Mid-Ordinate Rule given by:

$$(x_2 - x_1) = (y_2 - y_1) \phi[(y_1 + y_2)/2]$$

the function $\phi(y)$ in every case being evaluated as the reciprocal dx/dy of equation (1). The results are tabulated as computed depths at points where the theoretical profile has depth values of an integral number of feet. In Table 3 the results by the Mid-Ordinate method are shown in parentheses. The profiles were integrated from the control depth of 1 ft to a maximum depth of 5 ft, employing increments ranging from the maximum possible of 4 ft decreasing by successive halving, until reasonable accuracy had been attained. Even allowing for the fact that the Simpson's Rule method requires the evaluation of the function at half the stated increment, the comparison between the two methods is illuminating. In both cases a considerable increase in efficiency is obtained by the Simpson's Rule method, efficiency being measured in terms of the accuracy yielded in relation to the computational effort involved. It is apparent that the triangular channel produces a higher degree of curvature of the function dx/dy than in the semi-infinite rectangular case. It would seem that optimum step size could be related to the second derivative of the varied flow function, thus allowing the engineer to estimate in advance the required increment for any desired degree of accuracy in any set of circumstances.

118. Both in the Authors' review of computational methods and their novel contributions to integral surface profile equations, it is noticeable that the Chezy equation is used to describe the resistance to flow wherever this is at all reasonable.

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Table 3. Comparison of depths obtained by numerical integration using (i) Simpson's Rule and (ii) Mid-Ordinate Rule. (Latter values shown in parentheses)

Analysis of triangular channel

Distance, <i>ft</i>	Theoretical depth	Depth by numerical integration for different step sizes		
		<i>dy</i> = 4·0	<i>dy</i> = 2·0	<i>dy</i> = 1·0
0·00	1	1	1	1
– 1876·85	2			1·99731 (2·08827)
– 26 814·5	3		2·98654 (3·20418)	2·99909 (3·06652)
– 167 832·25	4			3·99960 (4·05223)
– 691 522·5	5	4·95570 (5·42278)	4·99670 (5·15310)	4·99979 (5·04271)

Table 3 (cont'd). Comparison of depths obtained by numerical integration using (i) Simpson's Rule and (ii) Mid-Ordinate Rule. (Latter values shown in parentheses)

Analysis of triangular channel

Distance, <i>ft</i>	Theoretical depth	Depth by numerical integration for different step sizes		
		<i>dy</i> = 0·5	<i>dy</i> = 0·25	<i>dy</i> = 0·125
0·00	1	1	1	1
– 1876·85	2	1·99983 (2·02576)	1·99999 (2·00672)	2·00000 (2·00170)
– 26 814·5	3	2·99995 (3·01795)	3·00000 (3·00458)	3·00000 (3·00115)
– 167 832·25	4	3·99998 (4·01366)	4·00000 (4·00345)	4·00000 (4·00087)
– 691 522·5	5	4·99999 (5·01099)	5·00000 (5·00277)	5·00000 (5·00069)

Table 3 (cont'd). Comparison of depths obtained by numerical integration using (i) Simpson's Rule and (ii) Mid-Ordinate Rule. (Latter values shown in parentheses)

Analysis of semi-infinite rectangular channel

Distance, f_t	Theoretical depth	Depth by numerical integration for different step sizes			
		$dy=4.0$	$dy=2.0$	$dy=1.0$	$dy=0.5$
0.0	1	1	1	1	1
-1077.24	2			1.99989 (2.06032)	1.99999 (2.01579)
-7955.52	3		2.99932 (3.16057)	2.99996 (3.04332)	3.00000 (3.01106)
-29 393.8	4			3.99998 (4.03342)	4.00000 (4.00846)
-79 032	5	4.99711 (5.37788)	4.99981 (5.10513)	4.99999 (5.02710)	5.00000 (5.00683)

Since many engineers make use of an exponential equation of the Manning-Strickler type, it is perhaps of interest to reference the contribution of Gunder²⁴ in integrating the equation of varied flow for this case. The resulting expression, unfortunately, makes even the Authors' equation (18) appear relatively simple, which perhaps explains the former's infrequent application.

119. Irrespective of whether the integration is analytic or arithmetic, the engineer is presented with a solution in which the distance x is the dependent variable, whereas in practice it is usual to require a value for the depth y at a specified distance. In my opinion there is advantage here also in adopting a numerical step-by-step technique, modified by simple interpolation. The iterative solution for y by means of one of the integrated equations is frequently bedevilled by instabilities as a result of different terms in the equation becoming dominant as the regime of the flow varies.

120. Turning now to the problem of flow profile integration in non-prismatic channels, the Authors have referenced a number of step-by-step solutions (e.g. references 3 and 4). It is quite common in such methods to use the approximation that the head loss is given by the product of the reach length and the arithmetic mean of the energy slopes at each end of the reach. Thus

$$E_2 - E_1 = (x_2 - x_1)(i_{r1} + i_{r2})/2$$

In my experience, it is more realistic to assume that the equivalent energy slope is given by the square root of the product of the two energy slopes. Thus

$$\Delta E = \Delta x \cdot \text{sign (larger of } i_{r1} \text{ and } i_{r2}) [\text{abs } (i_{r1} i_{r2})]^{1/2}$$

In steady flow this may safely be simplified to

$$\Delta E = \Delta x (i_{r1} i_{r2})^{1/2}$$

121. The validity of the approximation may be tested by assuming a non-prismatic reach of rectangular cross section in which both the depth and breadth of

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Fig. 12. Comparison of the three expressions for different values of α

the flow area increase linearly with respect to distance at a rate of $\alpha^{1/3}$. It may then be easily shown that

$$A_2 = \alpha A_1$$

$$p_2 = \alpha^{1/2} p_1$$

and

$$m_2 = \alpha^{1/2} m_1$$

Thus

$$A_2^2 m_2^{4/3} = \alpha^{8/3} A_1^2 m_1^{4/3}$$

Putting $A^2 m^{4/3} = F(y)$ and employing Simpson's Rule,

$$\Delta E = h_t = \text{coef.} \Delta x \cdot \frac{1}{6} \left[\frac{1}{\phi(y_1)} + \frac{4}{\phi(y_m)} + \frac{1}{\phi(y_2)} \right]$$

where $\phi(y_m)$ represents the function at the mid-section and is defined by the relation $y_m = y_1 (0.5 + \alpha/2)$. Substituting the relations involving the term α we obtain

$$\frac{h_t A_1^2 m_1^{4/3}}{\text{coef.} \Delta x} = \frac{1}{6} \left[1 + \frac{4}{(0.5 + \alpha/2)^{8/3}} + \frac{1}{\alpha^{8/3}} \right]$$

Assuming this to be a reasonable approximation to the integral, one may then compare the results of other numerical approximations. The simple product rule yields the equation

$$i_t = \frac{\text{coef.} \Delta x}{A_1 A_2 m_1^{2/3} m_2^{2/3}}$$

from which

$$\frac{h_t A_1^2 m_1^{4/3}}{\text{coef.} \Delta x} = \frac{1}{\alpha^{4/3}}$$

The alternative arithmetic mean, which is frequently used, may be shown to be equivalent to the expression

$$\frac{h_f A_1^2 m_1^{4/3}}{\text{coef. } \Delta x} = 0.5 \left(1 + \frac{1}{\alpha^{8/3}} \right)$$

122. Graphical comparison of the three expressions for different values of α is shown in Fig. 12, from which it is apparent that the product rule gives a measure of head loss quite close to that found by the Simpson's Rule method, over a wide range of values of α . In contrast the arithmetic mean values diverge rapidly from the others as the value of α exceeds about 1.5. Apart from any increased accuracy which may result, the device of the product rule may be employed to linearize the friction term in a dynamic equation in which the depth at one of the cross sections is to be determined.

123. In conclusion, I would like to enquire whether or not the Authors considered using the method of superposition first employed by Mononobe to obtain experimental profiles considerably in excess of the length of experimental facility available.

Dr Allen and Dr Enever

In reply to **Professor Jack Allen's** question regarding circular sections, we did not have the opportunity to investigate such sections experimentally, and it was not found practicable to spend the necessary time on a theoretical analysis.

125. Professor Allen's reference to the apparent paradox of the accurate result obtained by the late Professor A. H. Gibson, assuming a broad rectangular channel instead of the given circular section, may be explained as follows.

126. Consider the equation

$$\frac{dy}{dx} = \frac{i - i_t}{1 - V^2 t / (gA)},$$

where the notation is as given on page 784 of the Paper.

If we examine the denominator first,

$$\frac{V^2 t}{gA} = 0.0161 \text{ at } y = 5.48 \text{ ft}$$

and

$$\frac{V^2 t}{gA} = 0.035 \text{ at } y = 4.52 \text{ ft}$$

On the other hand

$$\frac{V^2}{gy} = 0.0156 \text{ at } y = 5.48 \text{ ft}$$

and

$$\frac{V^2}{gy} = 0.029 \text{ at } y = 4.52 \text{ ft}$$

127. All these values are so small compared with unity that any reasonable error in their estimation is unimportant. This is because the flow is well within the sub-critical region, since it may be calculated that $y_c = 0.258D = 1.88 \text{ ft}$, where D is the diameter of the section, which in this case is 7.30 ft.

Now consider the numerator

$$i - i_t = i \left(1 - \frac{i_t}{i} \right)$$

For the circular section

$$i = \frac{Q^2}{A_n^2 C^2 m_n}$$

where the suffix n denotes normal flow conditions.

Also

$$i_t = \frac{Q^2}{A^2 C^2 m}$$

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Therefore

$$\frac{i_t}{i} = \left(\frac{A_n}{A}\right)^2 \frac{m_n}{m}$$

For a broad rectangular section, the ratio

$$\frac{i_t}{i} = \left(\frac{y_n}{y}\right)^3$$

The area corresponding to the normal depth for the circular section is $A_n = 19.2$ sq. ft, and the hydraulic radius $m_n = 1.76$ ft.

At a depth of 5.48 ft

$A = 33.8$ sq. ft and $m = 2.21$ ft.

Hence

$$\frac{A_n^2 m_n}{A^2 m} = \left(\frac{19.2}{33.8}\right)^2 \times \frac{1.76}{2.21} = 0.258$$

whereas

$$\left(\frac{y_n}{y}\right)^3 = \left(\frac{3.40}{5.48}\right)^3 = 0.238$$

At a depth of 4.52 ft

$A = 27.1$ sq. ft and $m = 2.05$ ft.

Hence

$$\frac{A_n^2 m_n}{A^2 m} = \left(\frac{19.2}{27.1}\right)^2 \times \frac{1.76}{2.05} = 0.433$$

whereas

$$\left(\frac{y_n}{y}\right)^3 = \left(\frac{3.40}{4.52}\right)^3 = 0.426$$

Taking an intermediate depth of 5.00 ft

$A = 30.5$ sq. ft and $m = 2.15$ ft.

Hence

$$\frac{A_n^2 m_n}{A^2 m} = \left(\frac{19.2}{30.5}\right)^2 \times \frac{1.76}{2.15} = 0.324$$

whereas

$$\left(\frac{y_n}{y}\right)^3 = \left(\frac{3.40}{5.00}\right)^3 = 0.314$$

Hence in all three cases there is fairly good agreement between

$$\frac{A_n^2 m_n}{A^2 m} \quad \text{and} \quad \left(\frac{y_n}{y}\right)^3$$

128. There is therefore little error involved in the value of the numerator $i(1 - (i_t/i))$ if the channel is taken to be broad and rectangular.

129. It has already been demonstrated that the denominator $1 - V^2 t/gA$ is sensibly the same for the actual channel and for a broad rectangular one. Consequently we can replace the circular section by a broad rectangular section provided that our value of C is based upon the actual circular section, and provided that the channel is not nearly empty or nearly full.

130. The Authors are grateful to Mr Smith for his interesting and detailed contribution to the discussion. Since working on the project described in the Paper, one of the Authors has been engaged on developing and applying computing techniques to a range of problems in fluid mechanics at the British Hydromechanics Research Association. He would agree from this experience that certain standard procedures

Table 4

α	$\frac{h_r A_1^2 m_1^{4/3}}{\text{coef. } \Delta x}$		
	Simpson's Rule (A. A. Smith)	Simpson's Rule (as modified above)	Authors' exact integral
1	1.00	1.00	1.00
2	0.42	0.44	0.43
3	0.28	0.30	0.29
4	0.23	0.25	0.22
5	0.21	0.22	0.18

can be of great importance, and can be incorporated into a number of programs. He would also agree that a program for developing water surface profiles could quite easily be obtained in such a way. However we consider that, in spite of the ease with which computers can now be used, some investigation of the range of validity of analytical solutions may still be valuable, as a number of engineers still do not have immediate access to a computer.

131. As regards the step size selected, we have found that, with modern high speed computers, not much time is lost by choosing a time step for integration that is shorter than is strictly necessary, thus obviating the need for any detailed study of the relationship between the step size and the accuracy. That the accuracy is at least adequate can be checked by running the program with a smaller step size, as described in the Paper. However, it is realized that this will not give the most efficient use of computer time, and from this point of view we find Mr Smith's comments most interesting.

132. We would agree that numerical step-by-step methods have advantages when the depth at a specified distance up or downstream is required. Indeed, they need not be modified by interpolation, as an equation of the form $dy/dx = \phi(y)$ is merely a particular form of the equation $dy/dx = \phi(x, y)$, which can be integrated directly by the Runge-Kutta method to give y for specified intervals of x . In fact this was the method we used in our numerical work.

133. We also agree with Mr Smith in his assertion that a product rule gives a better value than an arithmetic mean for the friction slope to be used in head loss calculations. We would like to point out, however, that it was not necessary to use Simpson's Rule to evaluate the integral in the example Mr Smith chose to illustrate this point, as the function concerned can be integrated exactly. That is

$$\begin{aligned} \frac{h_r A_1^2 m_1^{4/3}}{\text{coef. } \Delta x} &= \int_0^1 \frac{1}{[1 + (\alpha^{1/2} - 1)x]^{16/3}} dx \\ &= \frac{3}{13(\alpha^{1/2} - 1)} \left(1 - \frac{1}{\alpha^{13/6}} \right) \end{aligned}$$

134. Over the range of x from 1 to 5 this function follows very closely the function obtained by Mr Smith using Simpson's Rule, thus supporting his preferences both for using Simpson's Rule for integration where appropriate and for using a product rule for friction slope. Incidentally, we believe that the depth at the mid-section in Mr Smith's analysis should have been $y_m = y_1(0.5 + \alpha^{1/2}/2)$, thus making the middle term of the Simpson's Rule equation $4/(0.5 + \alpha^{1/2}/2)^{16/3}$. However we appreciate that this will make only a small difference to the numerical results. Values of the function are tabulated in Table 4 using both Simpson's Rule integrals and our exact solution.

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135. In answer to Mr Smith's final question, we did consider the use of superposition to obtain longer profiles, but this could not have been employed to any great extent as the height of the channel walls would have limited the range of depths that could have been obtained in one extended run. Superposition could only have been employed in tests where the rate of change of depth was fairly small, and difficulty would have been encountered in deciding over what fixed length to work out the errors if some tests had been made over longer distances than others.

136. Finally, we wish to correct a misprint in equation (3) (p. 786). The last term in brackets should be

$$-2\sqrt{3} \tan^{-1} \left[\frac{\sqrt{3}y_n}{(2y+y_n)} \right]$$

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