**A mathematical analysis of the stress-distribution in cylindrical
concrete shell roofs**

by

Deryck Norman de Garrs Allen, M.A., Assoc.I.C.E.

and

John Michael Leigh Holman, B.Sc., Ph.D.

Mr W. J. Morris (Department of Civil and Municipal Engineering, University College, London) wrote that the results obtained by the Authors, and in particular the membrane stresses shown in Figs 7a and 9a were substantially different in form from the results of Gerrard⁴ and Gibson,⁵ and the experimental work at the Building Research Centre⁶ and the Cement and Concrete Association.⁷ They also differed markedly from the results from photoelastic⁸ and strain gauge tests being carried out at University College. Tests and calculations had been carried out on a large variety of shells, some with edge beams, some without, and under a variety of conditions imposed or assumed at the gable ends. The qualitative similarity of these results indicated that the assumed condition at the gable ends in this relaxation solution was unlikely to have a major effect on the stresses, particularly at mid span, but the major difference gave rise to a form of check which was totally independent of assumed boundary conditions.

FIG. 10

20. From the results given by the Authors, a simple analysis of forces and moments was carried out. The results for the longitudinal force concerned were shown in Fig. 10. The total longitudinal force on the shell changed from compressive at the mid span to a larger tensile force at the gable, which was inconsistent with a purely vertical 'dead weight' applied load.

21. The experiments so far carried out at University College led one to believe that the 'stress' T_x should not differ much from the prediction of simple beam theory and at mid span would have values from +820 at the edge to -400 at the crown. The Authors quoted +513 and -55.

22. Static equilibrium would seem to indicate an average error in T_x of the order of 50, and in working in from the edges where the boundary values were specified the error in mid span values might be expected to be subject to a cumulative error. Would the Authors comment on the accuracy of the final stress values, particularly having regard to the conditions of statics?

Mr R. D. Anchor (Chief Engineer, Smethwick Office, G.K.N. Reinforcements, Ltd) wrote that the Paper was to be commended in that it described a procedure for arriving at a numerical solution, whereas so many papers on shell roofs stopped short of the final answer. The Authors had unintentionally given their Paper a misleading title, and they would probably agree that their calculations applied to the distribution of stresses in an elastic shell. In a reinforced concrete shell the actual stresses were very appreciably affected by shrinkage, thermal effects and creep.

24. The Paper described very clearly a method of solving one of the differential equations which had been put forward as a solution to the problem of bending in cylindrical shells, but from the viewpoint of the practical designer it was not necessary to start from the differential equation. Papers by Tottenham⁹ and Bennett¹⁰ enabled all calculations to be completed in a few hours for internal and external cylindrical shells, with or without edge beams. The method arose from the use of an approximation first suggested by Schorer, which was considered by some investigators to be invalid. However, the method had been used over the last ten years to design many hundreds of shells which had actually been constructed and had not shown any signs of distress.

25. In particular, by using the empirical method described by Bennett, the arithmetic was reduced to a minimum and hence the possibility of an error occurring was also reduced.

26. The information concerning the stress distribution over the whole surface of the shell which was obtained by using the Authors' method was not normally necessary in a practical design. Furthermore, the examples which had been evaluated did not occur in practice. The Authors had understandably chosen the most simple case for their first example, but the deflexions of a shell of this span without edge beams would be unacceptable. The clamped condition at the end arches was also not realized in practice, except for internal spans.

27. Mr Anchor would like the Authors to explain a curious discrepancy which occurred in Fig. 7a. If the values of T_x were plotted for mid-span and at the end arch, the resultant forces of tension and compression at each section were not equal. This was valid, as the end arches were clamped and there was the possibility of a horizontal tensional force throughout the length of the shell. However, the resultant force at mid-span appeared to be a compression and the resultant force at the end arch was a tension of differing value.

28. It would be interesting to read the further Paper dealing with shells supported by edge beams, and it would be helpful if the Authors could indicate whether the method, as set out, could be adopted to deal with the case of vertical support but not horizontal or rotational restraint at the end arches.

29. Finally, had the Authors considered the problems involved in designing hyperbolic paraboloidal shells, and could their methods be used in this field?

Professor D. N. de G. Allen wrote in reply that Mr Morris and Mr Anchor had each drawn attention to a certain discrepancy in one of the two solutions presented in the Paper. The evidence for the discrepancy was a lack of balance in the resultant longitudinal force in the shell for which T_x values were given in Fig. 7a. There was

no doubt that this discrepancy existed and it was of importance to discover its cause. Further calculations had shown that the cause rested wholly in the inadequacy of some of the finite-difference approximations used in setting up the problem for numerical solution: that numerical solution in itself was highly accurate but errors in the actual equations solved arose, in particular, because the unusual variation, across the shell, of the longitudinal displacement u rendered the simple approximations used for $\partial u / \partial y$, on p. 412, and for $\partial^2 u / \partial y^2$, in the first of equations (15), insufficient.

31. By inclusion of difference correction terms this insufficiency could be removed. To check this assertion, calculations had been made of the effect of the necessary difference correction terms, and the results of this calculation were presented in terms of the resultant longitudinal force along the shell. Two of the values calculated by Mr Morris and used by him to draw the curve in Fig. 10 were incorrect: Fig. 11 gave his curve correctly and to twice the scale in the direction of the force co-ordinate.

FIG. 11

Points marked $\times$ were also plotted; they showed how the discrepancy displayed by the *variation* in the resultant force was accounted for by difference correction terms. The points marked were barely distinguishable from the curve in Fig. 11 and thus verified that the original omission of the difference correction terms was, for all practical purposes, entirely responsible for the lack of balance to which Mr Morris and Mr Anchor drew attention. Professor Allen would like to thank them for their most useful observation and regretted that the original solution was not realistic. Provided, however, that the correction terms were included, the theory given in the Paper remained valid and the method applicable.

32. To answer Mr Anchor's more general questions, first, the method could be adapted to deal with end arches which were supported vertically but not restrained horizontally or rotationally; the conditions at the arches, used in the Paper, that $u=0$ and $\partial w / \partial x=0$ would be replaced by conditions that $T_x=0$, i.e. $\partial u / \partial x=0$, and $G_x=0$, i.e. $\partial^2 w / \partial x^2=0$. Secondly, this method had not been applied to hyperbolic paraboloidal shells: the Paper aimed to show that the equations which had been put forward as governing the behaviour of a cylindrical shell could be solved by a numerical process. Since this work was undertaken, a different formulation of the basic theory of shells had been developed which it was hoped would prove not only more

amenable for numerical solution, but also more general in that it was applicable to shells of double curvature and of non-uniform thickness. Examples were being currently worked out and it was intended that they would, in due time, be published.

REFERENCES

4. F. A. GERARD. Ph.D.Thesis, University of London, May 1956.
 5. J. E. GIBSON. 'The design of cylindrical shell roofs'. Spon, London, 1961.
 6. G. R. MITCHELL. Shell roof research at the Building Research Station. *Proc. 2nd Symp. Concr. Shell Roof Construct.* Teknisk Ukeblad, Oslo, 1958, pp. 315-324.
 7. P. B. MORICE. Research on concrete shell structures. *Proc. Symp. Concr. Shell Roof Construct.* Cement & Concr. Ass., London, 1954, pp. 99-126.
 8. W. J. MORRIS. Photoelastic investigation of structural shells with particular reference to cylindrical shells. *Proc. Symp. Shell Res.* North Holland Publishing Co., Delft, August 1961.
 9. H. TOTTENHAM. A simplified method of design for cylindrical shell roofs. *Struct. Engr.*, vol. 32, 1954, pp. 161-180.
 10. H. D. BENNETT. Some recent developments in the design of reinforced concrete shell roofs. *Reinf. Concr. Rev.*, vol. 5, 1959, pp. 23-64.
-