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The analysis of grid frameworks of negligible torsional stiffness
by means of basic functions †

by

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Discussion

The following contributions were received in writing.

Mr J. E. Spindel (Senior Engineering Assistant, Bridge Design, London Midland Region, British Railways) observed that the method of using basic functions other than sine series which the Author had developed to solve distribution problems in torsionfree grid frameworks with various support conditions had the main merit of mathematical exactitude without being excessively complicated in the simple examples given in the Paper. That apparent neatness of solution might have led the Author to abandon sine series without fully exploiting them and thus to lose an easy way of considering torsion effects. Moreover, the basic functions which he used could become complicated in cases where continuous beams had unequal spans.

54. Approximate solutions for the examples quoted in the Paper, except for the cantilever, could be obtained by using sine series. The method of calculation used (a variation of one given by the Author⁵) would also give a close approximation to the fixing moment resulting from a point load applied at the centre of an infinitely wide isotropic slab.

55. Basically, all fixed or partially fixed ended beams (or strips in the case of a slab) were assumed to be replaced by simply supported longitudinal members of increased flexural rigidity so that the deflexions of the original member and its replacement were the same under the same sinusoidally distributed load. Only first and second harmonics were considered.

56. For a load $w(x) = w_n \sin \frac{n\pi x}{l}$ on a beam of flexural rigidity EI , the end slope was $w_n \frac{l^3}{n^3 \pi^3 EI}$. For the first harmonic that reduced to $w_1 \frac{l^3}{\pi^3 EI}$. The slope under a moment M applied at one end was $3 \frac{Ml}{EI}$. If one end was to be fixed, the fixing moment M_F became $\frac{w_1 l^3}{\pi^3 EI} \times \frac{3EI}{l}$. The deflexion of the beam was then $\left(1 - \frac{3\pi}{16}\right)$ times that of the same beam simply supported. The longitudinal stiffness of the simply supported member which was assumed to replace the fixed ended one must therefore be $EI / \left(1 - \frac{3\pi}{16}\right)$. The distribution of load was then calculated by any of the usual methods for a grid or slab with simply supported ends whose longitudinal members had that equivalent stiffness. Since fixed end moments depended on end slopes their distribution would follow that of the deflexions.

† Proc. Instn Civ. Engrs, vol. 6, p. 735 (Apr. 1957).

⁵ References 8 to 13 are given on p. 327.

57. Continuous grids or slabs could be treated as partially fixed at the supports, the amount of fixity being found by normal moment-distribution analysis. (The support moments left after distributing a fixed end moment taken as 100 gave the percentage fixity.) The actual support moments found by that process, after distribution, were then taken as loads on the unloaded span (which was again treated as partially fixed) and distributed once again.

58. The proportion of the "free" deflexion which was distributed to the outer girders had been calculated by that method for the examples given by the Author and was compared with the results he obtained and had since confirmed in Table 5.

TABLE 5

Case	Jaeger		Spindel
	Calc.	Exp.	
Encastré at both ends	0.0910	0.0910	0.0914
Propped cantilever	0.153	0.165	0.147
Two-span continuous girder } Loaded span .	0.162	0.162	0.141
	0.231	0.230	0.203

59. The greatest fixing moment arising from a point load W in the centre of an isotropic slab was found to be $0.168 W$ by that method, which compared with a value of $0.169 W$ calculated by Pucher⁸ and $0.172 W$ found experimentally by Kist and Bouma⁹.

60. The few good results given by Mr Spindel were not, of course, conclusive proof that sine series would provide a satisfactory solution in all cases, but did show that their use merited further investigation. Mr Spindel hoped to have an early opportunity of trying them for a continuous grid in which the longitudinal members had great torsional rigidity. There should be no difficulty in extending the use of the method to interconnected portal frames provided that there was no sway since those were, from the point of view of the analysis, merely continuous beams. Where there was sway, the analysis would indicate different sway of each portal—a condition likely to be resisted by an actual structure.

Mr Bernard Bramall (formerly Bridge Engineer, Railway and Bridge Section, Corporation of the City of Toronto) stated that the Author had carried the subject a valuable step forward in extending the analysis of grid frameworks by means of hyperbolic functions to the solution of continuous and cantilever spans.

62. So far as Mr Bramall could ascertain, the first publication of the method of applying harmonic analysis to single-span multiple girder bridges was his own contribution to the Paper by Gribble.¹⁰

63. Having since had occasion to use that elegant and powerful mathematical technique in a number of new bridge designs Mr Bramall dispelled forebodings concerning its imagined complexity. For design-office use he had employed Tables substantially the same as the Author's Tables of distribution coefficients for single spans, but with one significant difference; they represented only the transfer of loading, of bending, or of deflexion. They were in the nature of a correction to those functions which would be appropriate to the ordinary case calculated on the assumption that each main girder sustained unaided the load applied above it. Mr Bramall's Table 6, relating to a five-girder bridge, indicated the few changes required to transform the Author's Table (p. 751) into a Table of transfer factors. The particular advantage derived was that the series was rapidly convergent, even when the series for applied loading contained appreciable sec-

ondary or tertiary harmonics, and consequently the arithmetical work might be considerably reduced without sacrifice of accuracy.

64. Professor Inglis had remarked that deflexion curves were almost unaffected by other than the first few harmonic components of the loading, and in that particular study the shedding of load from one girder to its neighbours was effected entirely through the mechanism of differential deflexion. It was to be expected that the higher harmonics of any loading placed above a main girder would induce relatively little transfer to other girders; the transfer term became very small.

65. The contention might be illustrated by the example of a 95-ft five-girder bridge for which $\alpha = \frac{16 \cdot 715}{\nu^4}$.

Applying a uniformly distributed load of intensity w along the centre main girder, and expressing it as:

$$\frac{4w}{\pi} \left\{ \sin \frac{\pi x}{L} + \frac{1}{3} \sin \frac{3\pi x}{L} + \frac{1}{5} \sin \frac{5\pi x}{L} + \frac{1}{7} \sin \frac{7\pi x}{L} + \dots \right\}$$

It would be found that using the distribution coefficients the central bending moment became:

$$\frac{4wL^2}{\pi^3} \{ 0.3549 - 0.0311 + 0.0078 - 0.0029 + \dots \}$$

Using the modified transfer factors it was:

$$\frac{wL^2}{8} - \frac{4wL^2}{\pi^3} \{ 0.6451 - 0.0059 + 0.0002 - 0.0000 + \dots \}$$

$$= 0.0425 wL^2$$

It might be noted that taken to only the first term of the series the first method gave $0.0458 wL^2$, and the second method $0.0418 wL^2$.

66. Quite a good case could be made for expressing standard loadings in sinusoidal form and an example was given in Mr Bramall's contribution to the Proceedings.¹¹ It should be appreciated that the conventional E.U.D.L., whether supplemented by a point load or not, was itself an idealization intended to suit a specific mathematical approach. There could be little objection to an alternative idealization suiting the more general case of single-span grids just as well as the special case of statically determinate spans, provided that the same enveloping principle was observed.

TABLE 6.—TABLE OF TRANSFER FACTORS FOR FIVE-GIRDER BRIDGE

	Load on girder (1)	Load on girder (2)	Load on girder (3)
ρ_1	$\frac{(6 + 10\alpha)(5 + \alpha)\alpha}{(8 + 5\alpha)\Delta_2}$	(As in Paper)	(As in Paper)
ρ_2	(As in Paper)	$\frac{(184 + 228\alpha + 17.5\alpha^2)\alpha}{(8 + 5\alpha)\Delta_2}$	(As in Paper)
ρ_3	(As in Paper)	(As in Paper)	$-(8 + \alpha)4\alpha$
ρ_4	(As in Paper)	(As in Paper)	Δ_2 (As in Paper)
ρ_5	(As in Paper)	(As in Paper)	(As in Paper)

The Author, replying to Mr Spindel, first pointed out that simple examples had been used in the Paper for the sake of clarity of exposition; the method did not become unduly complicated in dealing with other structures.

68. The presentation of the Paper should not be taken to suggest a lack of vigour in exploiting sine waves in torsionally stiff grids. Professor Hendry and the Author had in

fact used sine waves extensively in torsionally stiff frames—in particular, for a method of analysing torsionally stiff continuous two-span grillages using sine waves (see reference 3).

69. Differentiating between sine waves on the one hand and “basic functions” on the other was, in the Author’s view, to lose the essential unity of treatment of grillages having various end conditions. As Mr Spindel had pointed out, the Author had himself in earlier work used the device of fictitious bending stiffness in conjunction with sine waves but had quickly abandoned that approach in favour of basic functions. It had seemed to him at that time (and still seemed) illogical to use the wrong shape and wrong bending stiffness to get an approximate answer when one could use the right shape and right bending stiffness and get a better answer. Again, statements such as “the deflexion of the beam is then x times the simply supported deflexion” inevitably gave rise to the question “deflexion at which point?” However, his main reason for abandoning the fictional bending stiffness approach was the question of accuracy of bending moments; it was fairly easy to get agreement on deflexions in such work, but that in itself was no guarantee that the bending moments would be even approximately correct.

70. Dealing with Mr Spindel’s Table 5, one could again see the falling-away of accuracy on the unloaded span of the continuous girder case; that falling-away was referred to in § 46 and seemed to be characteristic of methods in which the wrong shape was used as a vehicle of distribution.

71. The problem of interconnected portal frames (sway being prevented) had been solved by developing the appropriate basic functions, and details of that work would be published in the near future. The beams of the portals were treated as single-span girders with elastically restrained ends.

72. The Author was obliged to Mr Spindel for pointing out an inconsistency in the text. In § 43 in the evaluation of α the term $\frac{12}{(\nu\pi)^4} \left(\frac{36}{6}\right)^3 6 \times \frac{1}{2} \times \left(\frac{2}{3}\right)^3$ should be omitted. That term was the Author’s “quick estimate” of α based on the nominal sizes of cross-section of longitudinals and transversals; it gave a value of $\frac{23.65}{\nu^4}$. The correct value of α was found by determining EI and EI_T experimentally; that led to the value of $\frac{22.90}{\nu^4}$ quoted.

73. In reply to Mr Bramall, the Author believed it incorrect to refer to *the* method of harmonic analysis of grillages. Harmonic analysis in fact provided a genus within which several distinct methods had been developed. There was an extensive literature on grillage analysis, much of it pre-war in date, including some methods employing harmonic analysis (see for example references 7 and 12).

74. The question of whether one presented distribution data as transfer factors which added up to zero or as distribution coefficients which added up to unity was one of taste rather than of substance. A re-reading of the Paper would show that the technique of rapid convergence which Mr Bramall stressed had in fact been employed; thus in § 33: “the corresponding curves for the inner girder are conveniently arrived at . . . by subtracting from the free curves those components which have been distributed to the outer girders”. In the example given by Mr Bramall the free curve value was $\frac{wL^2}{8}$ and the terms which followed were precisely those components which had been distributed to the unloaded girders. The same point of rapid convergence was emphasized in reference 2. All the curves given in the Paper had in fact been prepared in that way. It would be foolish to obtain the curve for the loaded girder by summing the series for that girder, because of the slower convergence.

75. The Author agreed with Mr Bramall that a good case could be made out for expressing standard loadings in sinusoidal form and it was interesting to examine the influence on structural design which would result from such a change. For example, in

arches the adoption of a sinusoidal load leads directly to a sinusoidal shaped arch rib¹³, as compared with the parabolic shape which is a natural consequence of a uniformly distributed load.

76. He was obliged to Mr Bramall for pointing out a printing error in the table of distribution coefficients. In the five-girder bridge, with load on girder (2) the numerator of the p_3 term should begin with the numeral 224, not 22.

REFERENCES

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