TECHNICAL NOTE

Stress-dependency of elastic moduli for cross-anisotropic soils

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INTRODUCTION

Experimental evidence shows that the elastic Young's modulus E , the shear modulus G and the bulk modulus B of a soil element depend primarily on three factors: state of stress, density and stress history (Hardin & Black, 1969). By considering the principle of conservation of energy, Lade & Nelson (1987) expressed the Young's modulus of an isotropic elastic material as

$$E = Mp_a \left[\left(\frac{I_1}{p_a} \right)^2 + 6 \frac{1 + \nu}{1 - 2\nu} \frac{J_2}{p_a^2} \right]^\lambda \quad (1)$$

where I_1 is the first stress invariant, J_2 is the second deviatoric stress invariant, ν is Poisson's ratio, M and λ are material parameters, and p_a is atmospheric pressure. Comparison with experimental results from cubical triaxial tests on sand showed that equation (1) can capture the true non-linear elastic behaviour of soil (Lade & Nelson, 1987; Dakoulas & Sun, 1992). Equation (1) has been used to describe the elastic behaviour of both cohesionless and cohesive soils (Lade, 1990).

In reality, most natural soils display some degree of cross-anisotropy, having different elastic properties in the vertical and horizontal directions. Natural clay sediments, subjected to one-dimensional consolidation over long periods of time, form a fabric in which the particles are oriented horizontally, resulting in cross-anisotropic behaviour. The elastic behaviour of a cross-anisotropic material is defined by five independent elastic parameters (Wood, 1991). The ratio of the elastic Young's moduli in the horizontal and vertical directions, $n = E_h/E_v$, can be used as a measure of the degree of anisotropy. Typical

values of n for normally or lightly over-consolidated clays may range from 0.9 to 1.35 (Gazetas, 1982; Kirkpatrick & Rennie, 1972; Kirkgard & Lade, 1991). Heavily over-consolidated clays may develop a strong anisotropic behaviour, with reported typical values of n ranging from 1.35 to 2.4 (Ward, Samuels & Gutler, 1959; Atkinson, 1975; Gerrard, 1977), whereas desiccated clays have shown values of n from 1.8 to 4.2 (Gazetas, 1981). Anisotropic behaviour is also observed in varved clays and in sands with elongated particles oriented in the horizontal direction. The determination of the five elastic parameters requires advanced testing. To avoid the need for advanced equipment and complicated testing procedures, Graham & Houlsby (1983) assumed a certain form of the flexibility matrix of such soils and reduced the number of parameters from five to three, all of which can be easily derived from conventional tri-axial tests.

This Technical Note presents a general cross-anisotropic model for purely elastic behaviour based on five independent elastic parameters. In accordance with experimental evidence, the model assumes that the elastic Young's moduli and shear moduli are functions of the stress state; the three Poisson's ratios are practically constant. Moreover, the model assumes that, although E_h and E_v increase with increasing confining stress, the ratio E_h/E_v remains constant. This assumption is reasonable for soils with inherent anisotropy, for which there is no significant change in the material fabric with stress. This Technical Note deals with the effects of non-linearity and anisotropy on the elastic soil response.

STRESS-DEPENDENT ELASTIC MODULI

Figure 1 shows a system of co-ordinate axes (x , y , z) parallel to the principal axes of a cross-anisotropic material with identical elastic behaviour in the horizontal y , z directions, but different behaviour in the vertical x direction. The relationship between elastic strain increments $\Delta \epsilon_{ij}$ and effective stress increments $\Delta \sigma_{ij}'$ may be written using the generalized Hooke's law as shown in equation (2)

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$$\begin{Bmatrix} \Delta \varepsilon_x \\ \Delta \varepsilon_y \\ \Delta \varepsilon_z \\ \Delta \varepsilon_{xy} \\ \Delta \varepsilon_{yz} \\ \Delta \varepsilon_{zx} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_v} & -\frac{\nu_{vh}}{E_v} & -\frac{\nu_{vh}}{E_v} & 0 & 0 & 0 \\ -\frac{\nu_{vh}}{E_v} & \frac{1}{E_h} & -\frac{\nu_{hh}}{E_h} & 0 & 0 & 0 \\ -\frac{\nu_{vh}}{E_v} & -\frac{\nu_{hh}}{E_h} & \frac{1}{E_h} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2G_{hv}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1+\nu_{hh}}{E_h} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2G_{hv}} \end{bmatrix} \begin{Bmatrix} \Delta \sigma_x' \\ \Delta \sigma_y' \\ \Delta \sigma_z' \\ \Delta \tau_{xy} \\ \Delta \tau_{yz} \\ \Delta \tau_{zx} \end{Bmatrix} \quad (2)$$

where ν_{vh} is Poisson's ratio of strain in the horizontal direction caused by strain in the vertical direction, ν_{hh} is Poisson's ratio of strain in the horizontal direction caused by strain in the horizontal direction normal to the former, and G_{hv} is the shear modulus in any vertical plane.

For purely isothermal elastic behaviour, the net elastic work per unit volume generated or dissipated along a closed stress path C should be equal to zero. This condition suggests that the elastic work generated or dissipated between two different stress states is independent of the stress path. It can be shown that the increment of elastic work dW on any stress path can be expressed as

$$dW = \left(\frac{n}{m^2} - 1 \right) \frac{m\sigma_x'}{E_h} d(m\sigma_x') - \nu_{hh} \frac{I_1^*}{E_h} dI_1^* + 2(1 + \nu_{hh}) \frac{\sqrt{J^*}}{E_h} d(\sqrt{J^*}) \quad (3)$$

Fig. 1. Orientation of cross-anisotropy and stresses on a cubical element of soil

where

$$n = E_h/E_v \quad (4)$$

$$m = E_h \nu_{vh}/E_v \nu_{hh} \quad (5)$$

$$I_1^* = m\sigma_x' + \sigma_y' + \sigma_z' \quad (6)$$

$$J^* = \frac{1}{2}(m\sigma_x')^2 + \frac{1}{2}\sigma_y'^2 + \frac{1}{2}\sigma_z'^2 + (\alpha\tau_{xy})^2 + \tau_{yz}^2 + (\alpha\tau_{xz})^2 \quad (7)$$

$$\alpha^2 = E_h/2(1 + \nu_{hh})G_{hv} \quad (8)$$

Using Stokes' theorem, the integral along the closed path C can be transformed into an integral over the surface S bounded by the curve C in the space $(m\sigma_x', I_1^*, \sqrt{J^*})$. Thus

$$\begin{aligned} & \iiint_S \left[\left(\frac{\partial^2 W}{\partial(m\sigma_x') \partial I_1^*} - \frac{\partial^2 W}{\partial I_1^* \partial(m\sigma_x')} \right) n_1 \right. \\ & + \left(\frac{\partial^2 W}{\partial I_1^* \partial(\sqrt{J^*})} - \frac{\partial^2 W}{\partial(\sqrt{J^*}) \partial I_1^*} \right) n_2 \\ & + \left(\frac{\partial^2 W}{\partial(\sqrt{J^*}) \partial(m\sigma_x')} - \frac{\partial^2 W}{\partial(m\sigma_x') \partial(\sqrt{J^*})} \right) n_3 \, dS \\ & = 0 \end{aligned} \quad (9)$$

where (n_1, n_2, n_3) is a unit vector normal to S . Equation (9) is satisfied if $E_h = E_h(X)$, where

$$X = \frac{I_1^{*2}}{p_a^2} + 6 \frac{1 + \nu_{hh}}{1 - 2\nu_{hh}} \frac{J_2^*}{p_a^2} + \frac{3}{1 - 2\nu_{hh}} \left(\frac{n}{m^2} - 1 \right) \frac{(m\sigma_x')^2}{p_a^2} \quad (10)$$

$$J_2^* = \frac{1}{6}[(m\sigma_x' - \sigma_y')^2 + (\sigma_y' - \sigma_z')^2 + (\sigma_z' - m\sigma_x')^2] + (\alpha\tau_{xy})^2 + \tau_{yz}^2 + (\alpha\tau_{xz})^2 \quad (11)$$

I_1^* and J_2^* are the first invariant and the second

deviatoric stress invariant of the transformed stress tensor $\sigma^* = \{\sigma_x^*, \sigma_y^*, \sigma_z^*, \alpha\tau_{xy}, \tau_{yz}, \alpha\tau_{zx}\}$. Assuming that the Young's modulus is a power of X , E_h and E_v are expressed as

$$E_h = nMp_a \left[\left(\frac{I_1^*}{p_a} \right)^2 + 6 \frac{1 + v_{hh}}{1 - 2v_{hh}} \frac{J_2^*}{p_a^2} + \frac{3(n - m^2)}{1 - 2v_{hh}} \left(\frac{\sigma_x^*}{p_a} \right)^2 \right]^\lambda \quad (12)$$

$$E_v = Mp_a \left[\left(\frac{I_1^*}{p_a} \right)^2 + 6 \frac{1 + v_{hh}}{1 - 2v_{hh}} \frac{J_2^*}{p_a^2} + \frac{3(n - m^2)}{1 - 2v_{hh}} \left(\frac{\sigma_x^*}{p_a} \right)^2 \right]^\lambda \quad (13)$$

where M and λ are material constants. For isotropic soil, equations (12) and (13) reduce to equation (1). Also, for $\lambda = 0$ the model reduces to the linear cross-anisotropic model. More details about the derivation of equations (12) and (13) are given by Yu & Dakoulas (1993).

RESULTS

Figure 2 shows contours of constant E_v in the triaxial and π planes in a principal stress space ($\sigma_x^*, \sigma_y^*, \sigma_z^*$) for $n = m^2 = 0.5, 1$ and 2 and for six values of Poisson's ratio v_{hh} . (The meaningful part of these surfaces lies within the failure surface.) For isotropic soil (Fig. 2(b)) the contours of constant E_v are ellipsoidal surfaces with an axis of symmetry parallel to the hydrostatic axis, as shown by Lade & Nelson (1987). The interceptions of these surfaces with the triaxial plane are ellipses whose aspect ratio depends on v_{hh} ; their interceptions with the π plane are concentric circles. The ellipsoidal surface becomes a sphere for $v_{hh} = 0$, and degenerates to a line parallel to the hydrostatic axis as v_{hh} approaches 0.5 . For $E_h = 0.5E_v$ (Fig. 2(a)), the ellipsoidal surfaces change their aspect ratio and, as v_{hh} decreases, rotate their long axis on the triaxial plane towards the direction of σ_x^* . By contrast, for $E_h = 2E_v$ (Fig. 2(c)), as v_{hh} decreases, the ellipsoidal surfaces rotate their long axis towards the horizontal plane. If the results in Figs 2(a) and 2(c) are plotted in the principal stress space ($m\sigma_x^*, \sigma_y^*, \sigma_z^*$), the shape of the ellipsoids becomes identical to that in Fig. 2(b).

Figure 3 shows contours of constant E_v for $n = 0.5, 1$ and 1.5 , a constant value of Poisson's ratio $v_{hh} = 0.3$, and $m = 0.5\sqrt{n}, \sqrt{n}$ and $1.5\sqrt{n}$. As m increases the ellipsoidal surfaces of equal E_v increase in size and change their aspect ratio. When n increases the ellipsoidal surfaces change their aspect ratio and rotate their long axis on the triaxial plane.

APPLICATIONS: NON-LINEAR ELASTIC RESPONSE OF ANISOTROPIC SOIL

Effect of non-linearity

The effect of the stress-dependency of the elastic moduli on the stress-strain response is examined by performing an isotropic compression test on remoulded cross-anisotropic marine clay from the Gulf of Mexico. The cross-anisotropy of the clay is due to preconsolidation under K_0 conditions at a high stress level. The measured index properties and elastic parameters of the cross-anisotropic clay are shown in Table 1 (further details of the experimental programme are given by Yu & Dakoulas, 1993). After preconsolidation, the clay is unloaded to an isotropic stress level with $\sigma_1' = \sigma_2' = \sigma_3' = 100$ kN/m². Fig. 4 shows the confining stress σ_1' plotted against the elastic volumetric strain ϵ_v as σ_1' increases from 100 to 300 kN/m². The results in Fig. 4 show a non-linear stress-strain relationship, indicating that, for such levels of stress change, the dependence of the elastic moduli on the stress level is important.

Effect of cross-anisotropy

The effect of anisotropy on the non-linear soil response is examined by way of a number of triaxial compression and simple shear test simulations on marine clay specimens and on clay specimens with different degrees of anisotropy, expressed here through different values of n and α . The clay specimens are initially consolidated under K_0 conditions and then unloaded to $\sigma_1' = \sigma_3' = 100$ kN/m². Fig. 5 shows $q = \sigma_1' - \sigma_3'$ plotted against $p' = (\sigma_1' + 2\sigma_3')/3$ for four drained triaxial compression tests in which q increases from 0 to 50 kN/m² while $\sigma_3' = 100$ kN/m². It also shows q and the lateral strain ϵ_3 plotted against the axial strain ϵ_1 . The solid lines in Figs 5(b) and 5(c) (and in Figs 6(b) and 6(c)) represent the response of the marine clay ($n = 1.26$); the broken lines represent clay specimens with elastic properties as given in Table 1, but with n assumed equal to $0.75, 1$ and 2 . Due to the dependence of the elastic moduli on the stress

Parameter	Value
Liquid limit	74%
Plastic limit	30%
K_0	0.59
M	3.60
λ	1.29
v_{hh}	0.18
$n = E_h/E_v$	1.26
$m = E_h v_{hh}/E_v v_{hh}$	1.39
α	2.39

Table 1. Index properties and measured elastic parameters of cross-anisotropic marine clay from the Gulf of Mexico

Fig. 2. Contours of Young's modulus E_v in the triaxial and octahedral planes of the principal stress space for $n = m^2$: (a) $n = 0.5$; (b) $n = 1$; (c) $n = 2$

level, the stress-strain relationships are not straight lines, although, for the relatively small stress increment examined, the effect of non-linearity is small. As expected, the anisotropy parameter $n = E_h/E_v$ has a significant effect on the axial and lateral strains, which decrease sub-

stantially as n increases. Fig. 6 shows the undrained behaviour of the four anisotropic specimens examined, subjected to triaxial compression, in which the total vertical stress σ_1 was increased until $q = 50 \text{ kN/m}^2$, σ_3 was equal to 100 kN/m^2 . As q increases, the stress path for isotropic behav-

Fig. 3. Contours of Young's modulus E_v in the triaxial and octahedral planes of the principal stress space for various values of m ($v_{bh} = 0.3$): (a) $n = 0.5$, (b) $n = 1$, (c) $n = 2$

Fig. 4. Elastic non-linear stress-strain behaviour during isotropic compression of cross-anisotropic clay

Fig. 5. Elastic non-linear response during drained tri-axial testing of cross-anisotropic clay: (a) stress path; (b) deviator stress plotted against axial strain; (c) lateral strain plotted against axial strain.

Fig. 6. Elastic non-linear response during undrained tri-axial testing of cross-anisotropic clay: (a) stress path; (b) deviator stress plotted against axial strain; (c) lateral strain plotted against axial strain

behaviour is a vertical line, whereas the stress path direction tends to the right for $n < 1$ and to the left for $n > 2$, approaching the failure surface. Excess pore water pressure u develops at different rates for the four values of n . For $q = 50$ kN/m² the pore pressure increment for the marine clay ($n = 1.26$) is 2.2 times larger than the pore pressure assuming $n = 0.75$, and 1.42 times smaller than the pore pressure assuming $n = 2$. This substantial difference in the pore pressure development is also reflected in the significant deviation of the stress paths.

Figure 7 shows results from simulations of drained torsional shear tests on four anisotropic specimens corresponding to $\alpha = 0.5, 1, 2$ and 2.39 (marine clay); the rest of the elastic parameters

Fig. 7. Elastic non-linear response during torsional shear testing of cross-anisotropic clay: (a) stress path; (b) distortional stress-strain behaviour

are assumed to be identical to those in Table 1. Fig. 7(a) shows the general distortional stress $q = \sqrt{(3/2)}\tau_{vh}$, where τ_{vh} is the shear stress, plotted against p' . During all tests, q increases from 0 to 50 kN/m² and p' is equal to 100 kN/m². Fig. 7(b) shows q plotted against the distortional strain $\epsilon_q = \sqrt{(1/3)}\gamma_{vh}$ where γ_{vh} is the shear strain, for the four specimens. The results show the non-linearity of the stress-strain relationships and the substantial effect of α on the values of ϵ_q . Indeed, ϵ_q is equal to 0.18% for $\alpha = 0.5$, 0.68% for $\alpha = 1$, 2.23% for $\alpha = 2$ and 2.92% for $\alpha = 2.39$. Results from additional tests, not shown here, indicate that ϵ_q decreases with increasing values of n .

RELATIONSHIP TO GRAHAM-HOULSBY LINEAR MODEL

Graham & Houlsby (1983) presented a three-parameter linear model for cross-anisotropic soils, based on E_v , ν_{vh} and $n = E_h/E_v$, all of which can be derived from conventional triaxial tests. It can easily be verified that the general non-linear cross-anisotropic model presented in this Technical Note reduces to a non-linear extension of the Graham-Houlsby model, if it is assumed that

$$m = \sqrt{n} \quad (14)$$

$$\alpha = \sqrt[3]{n} \quad (15)$$

If it is also assumed that λ is zero, the model presented here reduces to the exact Graham-Houlsby linear model. Some experimental data on London clay (Gazetas, 1981) and the marine clay data from the Gulf of Mexico (Table 1) indicate that the assumptions (equations (14) and (15)) implied in the Graham-Houlsby linear model are not valid for those clays. However, the Graham-Houlsby linear model and its non-linear extension established here offer the significant advantage of using conventional triaxial testing to derive the three parameters. In the absence of data for an exact five-parameter model requiring more advanced cubical triaxial or hollow cylinder tests, it is definitely a better choice than an isotropic model.

CONCLUSIONS

A general model for the dependence of elastic moduli of a cross-anisotropic soil on the stress state has been presented, and the effects of non-linearity and anisotropy on the elastic soil response have been examined. The form of the elastic moduli is derived rigorously by considering the conservation of energy during a cycle of loading along an arbitrary closed stress path. The importance of the stress-dependence of the elastic moduli for the elastic behaviour of an anisotropic soil is elucidated by a parametric study, which examines the effects of the stress state, Poisson's ratios and degree of cross-anisotropy. The conclusions drawn from parametric studies using simulations of triaxial and shear tests are as follows.

The non-linear effect on the stress-strain behavior, due to stress-dependency of the elastic moduli, is important for large stress changes but could be ignored for small stress changes.

The effects of cross-anisotropy may be important for the elastic non-linear stress-strain behaviour of such soils. Substantial differences have been found in the stress-strain relationships and the development of excess pore water pressures for E_h/E_v ratios of 0.75–2. As a result, the approximation of the elastic behaviour of a strongly cross-anisotropic soil with an isotropic model, often adopted for simplicity in practice, may lead to substantial errors in the prediction of the soil response.

The model presented offers a general method for considering the non-linearity in the elastic soil behaviour caused by the dependency of the elastic moduli on the stress state for cross-anisotropic soils, and may constitute the elastic component of an elasto-plastic model.

For special values of two parameters, the model presented extends the approximate cross-

anisotropic linear model of Graham & Houlsby (1983) to a non-linear one, which requires only conventional triaxial tests.

NOTATION

E	Young's modulus of isotropic soil
E_v	Young's modulus in the vertical direction
E_h	Young's modulus in the horizontal direction
G	shear modulus of isotropic soil
G_{hv}	shear modulus in vertical plane
I_1, I_1^*	first invariant of standard and transformed stress tensors
J_2, J_2^*	second deviatoric invariant of standard and transformed stress tensors
M	material parameter
m	v_{hv}/v_{hh}
n	v_{vh}/v_{hh}
p_a	atmospheric pressure
p'	mean effective stress
q'	deviatoric stress
W	elastic work
α	$\sqrt{[E_h/2(1 + v_{hv})G_{hv}]}$
ε	strain
λ	material parameter
v_{hv}	Poisson's ratio of strain in vertical direction caused by strain on horizontal direction
v_{vh}	Poisson's ratio of strain in horizontal direction caused by strain on vertical direction
v_{hh}	Poisson's ratio of strain in horizontal direction caused by strain on orthogonal horizontal direction
σ'	effective normal stress
τ	shear stress

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REFERENCES

- Atkinson, J. H. (1975). Anisotropic elastic deformations in laboratory tests on undisturbed London clay. *Géotechnique* **25**, No. 2, 357–384.
- Dakoulas, P. & Sun, Y. (1992). Fine Ottawa sand: experimental behavior and theoretical predictions. *J. Geotech. Engng Div. Am. Soc. Civ. Engrs* **118**, No. 12, 1906–1923.
- Gazetas, G. (1981). *Deformational soil anisotropy: experimental evaluation and mathematical modeling*. Report 8102. Cleveland: Case Western Reserve University.
- Gazetas, G. (1982). Stress and displacement in cross-anisotropic soils. *J. Geotech. Engng Div. Am. Soc. Civ. Engrs* **108**, No. 4, 532–553.
- Gerrard, C. M. (1977). Background to mathematical modeling in geomechanics. The roles of fabric and stress-history. *Finite elements in geomechanics*, pp. 33–120. New York: Wiley.
- Graham, J. & Houlsby, G. T. (1983). Anisotropic elasticity of a natural clay. *Géotechnique* **33**, No. 2, 165–180.
- Hardin, B. O. & Black, W. L. (1969). Vibration modulus of normally consolidated clay. *J. Soil Mech. Fdns Div. Am. Soc. Civ. Engrs* **95**, No. 11, 1531–1537.
- Kirkgard, M. M. & Lade, P. V. (1991). Anisotropy of normally consolidated San Francisco Bay mud. *Geotech. Test. J.* **14**, No. 3, 231–246.
- Kirkpatrick, W. M. & Rennie, I. A. (1972). Directional properties of consolidated kaolin. *Géotechnique* **22**, No. 1, 166–169.
- Lade, P. V. (1990). Single-hardening model with application to NC clay. *J. Geotech. Engng Am. Soc. Civ. Engrs* **116**, No. 3, 394–414.
- Lade, P. V. & Nelson, R. B. (1987). Modelling the elastic behaviour of granular materials. *Int. J. Numer. Analyt. Meth. Geomech.* **11**, 521–542.
- Muir Wood, D. (1991). *Soil behaviour and critical state soil mechanics*. Cambridge: University Press.
- Ward, W. H., Samuels, S. G. & Gutler, M. E. (1959). Further studies of the properties of London clay. *Géotechnique* **9**, No. 2, 321–344.
- Yu, S. & Dakoulas, P. (1993). *Experimental investigation and constitutive modeling of marine clay*. Rice report to the Texas Advanced Technology Program. Houston TX: Rice University.