The double sliding, free rotating model for granular assemblies

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R. J. Evans, University of Washington, and R. E. Gibson, King's College, University of London

The Author is to be congratulated on his novel approach to this difficult subject. There are, however, two main points in the Note which we have difficulty in accepting.

First, inequalities (22) are certainly sufficient but surely not necessary to satisfy thermodynamic requirements. From (21) it is true that the condition

$$(a+b) \geq 0$$

is necessary for positive energy dissipation, but it is also sufficient and it does not seem necessary that a and b must separately be greater than zero; any combination satisfying the condition $(a+b) \geq 0$ is acceptable.

Second, inequality (23) which results from (22) is surely not invariant under co-ordinate rotation. It can certainly be shown that

$$(V_{x,x} \geq -V_{y,y}) \cos 2\psi + (V_{x,y} + V_{y,x}) \sin 2\psi \geq 0$$

which corresponds to (i) is invariant, but we believe that the inequality including the middle term of (23) is not invariant.

Author's reply to R. J. Evans and K. E. Gibson

I am grateful to Evans and Gibson for having raised these two points because my presentation of the double sliding, free rotating model was not satisfactory with respect to the first point and too short to cover the second. Both points, however, do not invalidate the essential result (the consecutive inequality (23)), which remains adequate in its presented form.

With respect to the first point I agree that the requirement of positive total energy dissipation only provides

$$(a+b) \geq 0$$

and this does not imply the more stringent condition (22), which is

$$a \geq 0$$

$$b \geq 0$$

In the Note, instead of treating the total energy, I should have pointed out that the frictional character of the sliding mechanism guarantees that every little bit of energy, wherever dissipated, is positive. The thermodynamic requirement of positive energy dissipation can therefore be stated in the more special form that the energy dissipated by each sliding mechanism separately is positive. Therefore, the reasoning beginning on the fourth line from the bottom of p. 160, up to equation (22) would have been more appropriate, if it had been presented in the following manner.

Transition to a Greek co-ordinate system ξ^α shows that ϵ_{ij} is a weighed tensor, because

$$\epsilon_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^i} \frac{\partial \xi^\beta}{\partial x^j} = \frac{\partial \xi^\alpha}{\partial x} \frac{\partial \xi^\beta}{\partial y} - \frac{\partial \xi^\alpha}{\partial y} \frac{\partial \xi^\beta}{\partial x} = J \cdot \epsilon_{ij}$$

Transformation of the components of the velocity gradient is given by

$$V_{i,k} = V_{\lambda,\gamma} \frac{\partial \xi^\lambda}{\partial x^i} \frac{\partial \xi^\gamma}{\partial x_k}$$

and for the stresses

$$\sigma_{jk} = \sigma_{\mu\nu} \frac{\partial \xi^\mu}{\partial x^j} \frac{\partial \xi^\nu}{\partial x^k}$$

By transition from Roman co-ordinates to Greek co-ordinates the foregoing expression for M is transformed as follows

$$\begin{aligned} M &= -\frac{1}{2} \epsilon_{ij} (V_{i,k} + V_{k,i}) \sigma_{jk} \\ &= -\frac{1}{2} J^{-1} \epsilon_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^i} \frac{\partial \xi^\beta}{\partial x^j} (V_{\lambda,\gamma} + V_{\gamma,\lambda}) \frac{\partial \xi^\lambda}{\partial x^i} \frac{\partial \xi^\gamma}{\partial x^k} \sigma_{\mu\nu} \frac{\partial \xi^\mu}{\partial x^j} \frac{\partial \xi^\nu}{\partial x^k} \\ &= -\frac{1}{2} J^{-1} \epsilon_{\alpha\beta} (V_{\lambda,\gamma} + V_{\gamma,\lambda}) \sigma_{\mu\nu} \left(\frac{\partial \xi^\alpha}{\partial x^i} \frac{\partial \xi^\lambda}{\partial x^i} \right) \left(\frac{\partial \xi^\beta}{\partial x^j} \frac{\partial \xi^\mu}{\partial x^j} \right) \left(\frac{\partial \xi^\gamma}{\partial x^k} \frac{\partial \xi^\nu}{\partial x^k} \right) \\ &= -\frac{1}{2} J^{-1} \epsilon_{\alpha\beta} (V_{\lambda,\gamma} + V_{\gamma,\lambda}) \sigma_{\mu\nu} \delta^{\alpha\lambda} \delta^{\beta\mu} \delta^{\gamma\nu} \\ &= -\frac{1}{2} J^{-1} \epsilon_{\alpha\beta} (V_{\alpha,\gamma} + V_{\gamma,\alpha}) \sigma_{\beta\gamma} \end{aligned}$$

The expressions for M in Roman and Greek co-ordinates are similar, except for the presence of the factor J^{-1} .

For rotation J is equal to $+1$, and complete identity is obtained showing invariance of the expression for M in that case.

For reflexion J is equal to -1 and M changes sign. This sign inversion does not spoil the invariance, because an introduction of equations (12) and (14) shows that

$$M = (-a+b) \sin \phi \cos \phi$$

Sign inversion amounts to the interchange of the shear strain rates a and b , which correspond to the S_1 and S_2 characteristic directions respectively. This interchange is a consequence of the reflexion which corresponds to the transition from a right-handed co-ordinate system to a left-handed one.

The original inequalities expressed by (12) and (14) were

$$-(a+b) \sin \phi \cos \phi \leq (-a+b) \sin \phi \cos \phi \leq (a+b) \sin \phi \cos \phi$$

In the reflected system they are

$$-(a'+b') \sin \phi \cos \phi \leq (a'-b') \sin \phi \cos \phi \leq (a'+b') \sin \phi \cos \phi$$

Both expressions guarantee the same thing; namely

$$0 \leq b; \quad 0 \leq a$$

$$0 \leq a'; \quad 0 \leq b'$$