The Authors are glad that Gadsby also obtained a ratio of $N_c/N_s = 1.5$ with his sleeved dynamic cone. It agrees with the ratio given in the Paper for shallow depths. The Authors have further observed that it may be difficult to push the sleeved cone in dense sands. In one study it was difficult to penetrate even a simple cone and the blow count was about 60–70 over each 30 cm penetration.

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FULLY DEVELOPED PLASTIC SHEAR FLOW OF GRANULAR MATERIALS

(MANDL, G. & FERNÁNDEZ LUQUE, R. (1970). *Géotechnique* **20**, No. 3, 277–307)

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This Paper is a valuable contribution to the understanding of the flow of granular materials, and clarifies a number of points. In particular the discussion of non-coaxiality of stress and strain-rate tensors gives a new and illuminating approach to this long-standing problem in the mechanics of granular materials.

Of particular interest to the Writer is the analysis in Appendix 2 of the model of a granular material proposed by de Josselin de Jong (1958, 1959). The Authors show that this model leads to the equation (equation (63a) of the Paper¹)

$$\left(\frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial y}\right) \sin 2\theta - \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) \cos 2\theta - \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} - 2\frac{\partial \theta}{\partial t}\right) \sin \rho = 0 \quad (1)$$

where v_x and v_y are velocity components referred to rectangular Cartesian co-ordinates (x, y) , θ is the angle of inclination of the largest compressive principal stress axis to the x axis and ρ is the angle of internal friction, here assumed constant. As the Authors point out, this equation differs slightly from one proposed by Spencer (1964), which in the present notation is

$$\left(\frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial y}\right) \sin 2\theta - \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) \cos 2\theta - \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} - 2\frac{D\theta}{Dt}\right) \sin \rho = 0 \quad (2)$$

where $D\theta/Dt$ denotes the convected derivative

$$\frac{D\theta}{Dt} = \frac{\partial \theta}{\partial t} + v_x \frac{\partial \theta}{\partial x} + v_y \frac{\partial \theta}{\partial y} \quad (3)$$

Equations equivalent to equations (2) and (3) have been given earlier by Mandel (1947).

The purpose of this discussion is to draw attention to the fact that equation (1) is not a possible constitutive relation for any material because it fails to satisfy the essential and fundamental physical requirement of being independent of a superimposed uniform rigid body translational motion. Thus suppose that

$$\left. \begin{aligned} v_x &= f(x, y, t) \\ v_y &= g(x, y, t) \\ \theta &= h(x, y, t) \end{aligned} \right\} \quad (4)$$

satisfy equation (1). Then equation (1) should also be satisfied by any velocity field which

¹ An error in the Paper in the sign of $\partial \theta / \partial t$ has been corrected.

differs from (4) only by a superimposed rigid body translation. Such a field is

$$\left. \begin{aligned} v_x &= f(x', y', t) + V_x \\ v_y &= g(x', y', t) + V_y \\ \theta &= h(x', y', t) \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where

$$\left. \begin{aligned} x' &= x - V_x t \\ y' &= y - V_y t \end{aligned} \right\} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \quad (6)$$

and V_x and V_y are constants. It is easily shown by substitution that (5) does not in general satisfy equation (1), and so equation (1) is necessarily invalid on physical grounds. On the other hand if (4) satisfy equation (2), then (5) always satisfy equation (2) and thus equation (2) is, from this point of view, an acceptable possible constitutive relation.

The source of the discrepancy lies in an implicit assumption in the analysis of the Paper that the slip directions (stress characteristics) are straight lines. This restriction may be removed in several ways; the following is a straightforward geometrical demonstration. De Josselin de Jong's theory postulates deformation by two superposed simple shearings on the stress characteristics. In steady motion $\partial\theta/\partial t=0$ and on a typical straight characteristic inclined at an angle δ to the x axis the deformation of an initially rectangular element of material in one such shear motion is shown in Fig. 1. Referred to x, y co-ordinates, the rate of strain components $\dot{\epsilon}_{xx}$, $\dot{\epsilon}_{yy}$ and $\dot{\epsilon}_{xy}$ and the rotation rate $\dot{\omega}$ for such a deformation are as shown in Appendix 2 of the Paper

$$\left. \begin{aligned} \dot{\epsilon}_{xx} &= -\dot{\epsilon}_{yy} = \frac{1}{2}\gamma \sin 2\delta \\ \dot{\epsilon}_{xy} &= -\frac{1}{2}\gamma \cos 2\delta \\ \dot{\omega} &= -\frac{1}{2}\gamma \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where γ is the rate of shear and the sign conventions are as in the Paper. Superposing two motions of this type on lines inclined at δ_1 and $\delta_2 + \frac{1}{2}\pi$ to the x axis, with respective shear rates γ_1 and $-\gamma_2$, and eliminating γ_1 and γ_2 from the resulting expressions for ϵ_{xx} , ϵ_{xy} and $\dot{\omega}$, gives, as in the Paper, equation (1) with $\partial\theta/\partial t=0$.

However, suppose that the characteristics are curved, and that the deformation consists of two superposed curvilinear shearing motions of the type shown in Fig. 2. In general a single shearing motion of this kind cannot be derived from a continuous velocity field, but a superposition of two such motions is possible. From Fig. 2 it is evident that, in addition to the rotation rate $-\frac{1}{2}\dot{\gamma}$ arising from the shearing deformation, the element also rotates due to its motion along the curved characteristic. The rate of this rotation is $v_1 \partial\theta/\partial S_1$, where v_1 is

the velocity in the slip direction and S_1 denotes distance along the characteristic. Thus the rotation rate for the curvilinear shearing motion is given by

$$\dot{\omega} = -\frac{1}{2}\gamma + v_1 \partial\delta/\partial S_1$$

When two such deformations are superposed, with shear rates γ_1 and $-\gamma_2$ on curves inclined at δ_1 and $\delta_2 + \frac{1}{2}\pi$ to the x axis, the total strain rate and rotation rate components become

$$\left. \begin{aligned} \dot{\epsilon}_{xx} &= -\dot{\epsilon}_{yy} = \frac{1}{2}\gamma_1 \sin 2\delta_1 + \frac{1}{2}\gamma_2 \sin 2\delta_2 \\ \dot{\epsilon}_{xy} &= -\frac{1}{2}\gamma_1 \cos 2\delta_1 - \frac{1}{2}\gamma_2 \cos 2\delta_2 \\ \dot{\omega} &= -\frac{1}{2}(\gamma_1 - \gamma_2) + v_1 \frac{\partial\delta_1}{\partial S_1} + v_2 \frac{\partial\delta_2}{\partial S_2} \end{aligned} \right\} \dots \dots \dots (8)$$

Eliminating γ_1 and γ_2 from (8), and using (equations (62) of the Paper)

$$\delta_1 = \theta - \frac{1}{4}\pi + \frac{1}{2}\rho$$

$$\delta_2 = \theta - \frac{1}{4}\pi - \frac{1}{2}\rho$$

together with the relation

$$v_1 \frac{\partial\theta}{\partial S_1} + v_2 \frac{\partial\theta}{\partial S_2} = v_x \frac{\partial\theta}{\partial x} + v_y \frac{\partial\theta}{\partial y}$$

gives equation (2) with $\partial\theta/\partial t = 0$. For the case of non-steady motion, the term involving $\partial\theta/\partial t$ in equation (2) may be recovered by referring the motion to rotating axes in the manner described in Appendix 2 of the Paper. In the special case in which the characteristics are straight, so that $\partial\theta/\partial x = \partial\theta/\partial y = 0$, equation (2) reduces to equation (1).

This slight modification gives equations free from the physical objections to equation (1) and the equations based on de Josselin de Jong's model become identical to those proposed by Spencer (1964). An essentially equivalent procedure for formulating equation (2) is to refer the motion to axes which rotate and translate with a typical element of the material, in the manner used by Spencer (1964).

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